Plane Trigonometry

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PREFACE TO THE SECOND EDITION

Whoever ventures to write a text book on a subject gives almost invariably some words of explanation for the appearance of 'yet another' in a field already overpopulated. And I also had my own. Yes, I was aware of a number of suitable books on elementary trigonometry that could sufficiently delineate the pattern which was desired. Why, then, was the addition ? This was because I believed that no single book on any subject could satisfy the needs and the desires of all the individuals of heterogeneous interests. Again, since no book is a true copy of its forerunners each has some characteristic merits of its own. And that gives it the right to exist. In my case, however, this was not the only reason. I had, to be frank, some weakness for trigonometry, the gateway to higher mathematics. I, therefore, fell naturally inclined to guide the uninitiated ones in the field by the direct but easy route to the gateway lest they get tired and miss the fascinating vistas of advanced mathematics. I was, however, hesitant, particularly when I had relatively more of efferves- cence than experience and I was afraid lest I misguide. Now that the second edition of the book is ready from the press, I feel relieved that my work could at least fulfil the require- ments of some individuals, guide, to the extent possible, some new entrants and thus merit its appearance.

This edition is practically a reprint of its predecessor retaining the characteristic features unimpaired. Few misprints have been corrected. Some minor additions and alterations have beed made here and there in the interest of integrity of the contents. In preparing this revision I have constantly referred to many letters sent by colleagues and students of the different colleges of Bengal and Assam. For this act of theirs, which bears testimony to the generosity of their heart, I shall remain ever grateful to all of them, and wish that it be continued in future. Some of my friends suggested to curtail the bulk of the book by dropping the contents which form the appendix. I am sorry I could not comply with their suggestion

for the sake of the completeness of the work. And it is not a difficult task to stop learning anywhere the reader wishes.

I shall be, as in the past, only too glad to receive, acknowledge and attend all types of criticisms, comments and suggestions for the future improvement of the work.

Indian Statistical Institute

A. B. Gupta

PREFACE TO THE FIRST EDITION

In this work the elements of Trigonometry have been presented in a manner suitable for beginners. It is, therefore, hoped that it would satisfy the requirements of the Pre-university, Higher Secondary and Technical students of different Indian Universities and Boards of Secondary Education. The subject-matter has been developed systematically and with conciseness without, however, giving up the necessary rigour. Much thought has been spent on the general arrangement and on the selection of suitable examples to illustrate the text. A large number of examples, over 750 in all, has been given and they are all graded to the extent possible. With the aid of the numerous worked-out examples the students may push on rapidly without much difficulty. The appendix at the end discusses a number of topics not covered by the syllabus of the pre-university and the allied courses. This has been done only for the sake of the completeness of the book. We have incorporated a large number of examples set in different examinations of various universities to give an idea of the standard of examination.

A list of important formulae and useful data and a subject index have been given for ready reference.

In preparing the book the author has been largely influenced by most of the authors—both Indian and English—of the standard works on the subject. To all of them he acknowledges his indebtedness.

The author would deem it a favour if criticisms and suggestions towards the improvement of the book are received.

Indian Statistical Institute **A. B. Gupta.**

Contents

Notations

α	(alpha)	π	(pai)
β	(beta)	ϕ	(phai)
γ	(gamma)	Σ	(sigma)
θ	(theta)	$\triangle$	(delta)

C. U. : Intermediate Examination, Calcutta University.

C. P. : B. Sc. (Pass), Calcutta University.

B. H. U. : Intermediate Examination Banaras Hindu University.

P. U. : „ Patna University.

A. U. : „ Allahabad University.

IMPORTANT FORMULÆ

I. Trigonometrical ratios

$\sin^2\theta+\cos^2\theta=1 \qquad \frac{\sin\theta}{\cos\theta}=\tan\theta$

$\sec^2\theta=1+\tan^2\theta$

$\operatorname{cosec}^2\theta=1+\cot^2\theta \qquad \frac{\cos\theta}{\sin\theta}=\cot\theta$

II. Compound angles

$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\pm\tan A\tan B}$$

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

III. Transformation of products

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\sin B=\sin(A+B)-\sin(A-B$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$2\sin A\sin B=\cos(A-B)-\cos(A+B)$

IV. Transformation of sums

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

V. Multiple angles

$\sin 2A = 2 \sin A \cos A.$ $\quad 2 \sin^2 A = 1 - \cos 2A$

$\cos 2A = \cos^2 A - \sin^2 A$ $\quad 2 \cos^2 A = 1 + \cos 2A$

$= 2 \cos^2 A - 1$ $\quad \tan^2 A = \dfrac{1 - \cos 2A}{1 + \cos 2A}$

$= 1 - 2 \sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$ $\quad \sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

$\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

$\sin 3A = 3 \sin A - 4 \sin^3 A$

$\cos 3A = 4 \cos^3 A - 3 \cos A$

$\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

VI. General values

If $\sin \theta = \sin \alpha$, $\quad \theta = n\pi + (-1)^n \alpha$

If $\cos \theta = \cos \alpha$, $\quad \theta = 2n\pi \pm \alpha$

If $\tan \theta = \tan \alpha$, $\quad \theta = n\pi + \alpha$

If $\sin \theta = 0$ or $\tan \theta = 0$, $\theta = n\pi$

If $\cos \theta = 0$, or $\cot \theta = 0$, $\theta = (2n+1)\pi/2$

VII. Inverse circular functions

$\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

$\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$

$\sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}$

$\tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \dfrac{x \pm y}{1 \mp xy}$

$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \dfrac{x + y + z - xyz}{1 - xy - yz - zx}$.

$\sin^{-1} x \pm \sin^{-1} y = \sin^{-1} (x\sqrt{1 - y^2} \pm y\sqrt{1 - x^2})$

$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1} \{xy \mp \sqrt{(1 - x^2)(1 - y^2)}\}$

VIII. Properties of a triangle

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$a = b \cos C + c \cos B$

$\tan \dfrac{B - C}{2} = \dfrac{b - c}{b + c} \cot \dfrac{A}{2}$

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{(s-b)(s-c)}{\Delta}$$

$$\Delta=\tfrac{1}{2}\,bc\sin A=\tfrac{1}{2}\,ca\sin B=\tfrac{1}{2}\,ab\sin C$$
$$=\sqrt{s(s-a)(s-b)(s-c)}$$
$$=\tfrac{1}{4}\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}$$
$$=\frac{abc}{4R}$$

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2\Delta}{bc}$$

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}$$

$$r=\frac{\Delta}{s}=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$
$$=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$$

$$r_1=\frac{\Delta}{s-a}=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=s\tan\frac{A}{2}$$

$$r_2=\frac{\Delta}{s-b}=4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}=s\tan\frac{B}{2}$$

$$r_3=\frac{\Delta}{s-c}=4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}=s\tan\frac{C}{2}$$

Useful Data

1 radian $=57°17'44{\cdot}8''$ nearly	$\pi^2=9{\cdot}87$
1 degree $={\cdot}0174533$ radians nearly	$\sqrt{2}=1{\cdot}414$
$\pi=\frac{22}{7}=3{\cdot}1416$ (approx)	$\sqrt{3}=1{\cdot}732$

circumference $=\pi\times$ diameter

Important identities

If $A+B+C=\pi$,

$\sin A+\sin B+\sin C=4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$
$\cos A+\cos B+\cos C=1+4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$
$\tan A+\tan B+\tan C=\tan A.\tan B.\tan C$
$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$
$\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$
$\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$
$\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

1

Measurements of angles

1-1. Trigonometry, in a sense, is the foundation of higher mathematics and in every branch of it—pure or applied—a knowledge of trigonometry is of paramount importance. It is divided into two parts—plane trigonometry and spherical trigonometry. In these pages we shall deal with the plane trigonometry only, the primary object of which was originally the measurement of plane triangles, that is to say, the establishment of the relations between the sides, angles and area of a triangle. But now the range of its application has extended and it carries on investigation relating to any angle, not necessarily connected with a triangle.

1-2. Generation of angles in trigonometry

The concept of an angle in trigonometry is more general than that in geometry, as will be clear from the discussion that follows.

Let a line OP starting from the position OA turn round O in the counter-clockwise direction. As it turns, it generates the angle AOP. The *amount of turning* undergone by OP measures the *angle* between OA and OP. Fig. 1 represents a particular position of the *generating line* OP when $\angle$AOP is acute.

In the process of revolution, when OP reaches the position OB, it traces out a right angle. An angle equal to two right angles will be generated when OP will come to the position OA′. As OP goes on revolving, it may reach the position as indicated

in Fig. 2 when the angle described would be greater than two right angles.

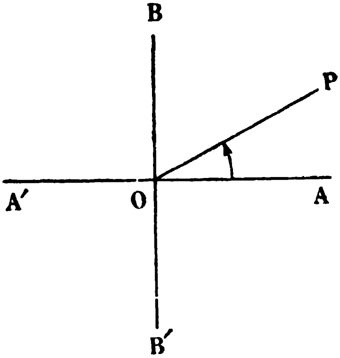

Fig. 1

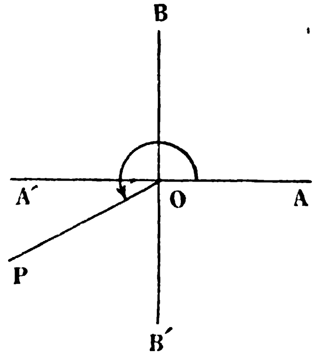

Fig. 2

As OP continues its revolution, it will reach the position OA whence it started. A complete revolution will thus be made and four right angles would be described.

The turning may go on indefinitely and an *angle of any magnitude* may be generated. Each time a complete revolution occurs, four right angles are traced.

Instead of counter-clockwise revolution, we could have made OP turn clockwise as well and a new set of angles would have been formed thereby.

Angles generated by counter-clockwise revolution are conventionally called *positive* and those generated by clockwise revolution *negative*.

Thus, in trigonometry are included angles of all magnitude positive and negative. Contrary to this, in Geometry, where angles are produced not by the rotation of a line about one of its extremeties, but by the intersection of two straight lines, only positive angles less than four right angles bear significance.

1-3. Measurement of angles

To measure an angle, as in all physical measurements, we, must first decide upon some *unit angle* ; then any angle will be expressed by the number of times it contains the unit angle. Depending on how we fix upon unit angle, there will be different systems of measuring angles.

Sexagesimal system : The unit of angle used in this system is the *degree*. A degree is defined to be the one-ninetieth part of a right angle. To avoid using fractions of a degree, the degree is again sub-divided into sixty equal parts called *minutes*, and each minute into sixty *seconds*. Thus

1 rt. angle	**$=90^\circ$ (degrees)**
1°	**$=60'$ (minutes)**
$1'$	**$=60''$ (seconds)**

In *numerical* calculations the sexagesimal measure is always used.

Circular system : The unit of angle used in this system of measurement is called a *radian* and the magnitude of any other angle is expressed by the number of times it contains the radian. This system is convenient for theoretical use.

A *radian* is defined to be the angle subtended at the centre of any circle by an arc whose length is equal to the radius of the circle.

In the adjoining figure, ABC is a circle whose centre is O. The arc BC is equal in length to the radius of the circle. By definition, therefore, the angle BOC is 1 radian.

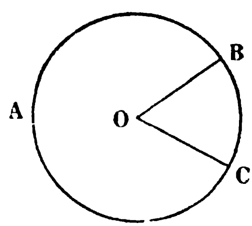

Fig. 3

We shall presently show that a radian is of constant magnitude independent of the particular circle used.

1-4. An important property of the circle : *The circumferences of circles are to one another as their diameters.*

Let ABC, A′B′C′ be any two circles of radii r and r'. In

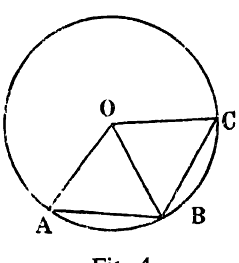

Fig. 4

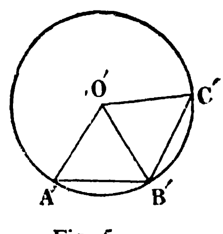

Fig. 5

each circle let a regular polygon of n sides be inscribed. Join the centres O, O′ to the vertices of the respective polygons.

Since the $\triangle^s$ OAB, O'A'B' are similar,

$$\therefore \frac{OA}{O'A'}=\frac{AB}{A'B'}=\frac{n.AB}{n.A'B'}=\frac{p}{p'},$$

where p and p' are the perimeters (i.e., sum of the sides) of the polygons.

$$\text{Or, } \frac{r}{r'}=\frac{p}{p'}. \quad (\because OA=r,\ O'A'=r')$$

This is true whatever be the number of sides in the polygons. So by taking n indefinitely large, the circumferences may in the limit be made coincide with the perimeters of the polygons.

$$\therefore \frac{r}{r'}=\frac{\text{circumference of circle ABC}}{\text{circumference of circle A'B'C'}}$$

As diameter is twice the radius, *circumference/diameter* is a constant quantity in all circles.

This constant is denoted by the Greek letter π and is incommensurable. It can only by obtained in the form of an infinite non-recurring decimal and is 3·14159 to the first five places of decimals. Approximately, $\pi=22/7$. A more correct value is 355/113.

1-5. If r be the radius of circle, we may write

$$\boxed{\text{circumference}=2\pi r} \qquad \text{............(1)}$$

1-6. To prove that a radian is an angle of constant magnitude

Let O be the centre of a circle of radius r (=OA). Let the

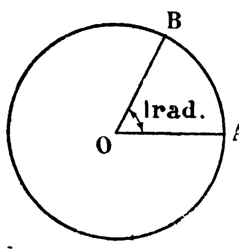

Fig. 6

arc AB be measured equal in length to OA. Join OB; then $\angle$AOB is a radian.

Now, since from geometry, the angles at the centre of a circle are proportional to the arcs on which they stand, we have

$$\frac{\angle AOB}{4 \text{ rt. angles}} = \frac{\text{arc } AB}{\text{whole circumference}}$$

or, $$\frac{1 \text{ radian}}{4 \text{ rt. angles}} = \frac{r}{2\pi r}$$

or, $1 \text{ radian} = \frac{2}{\pi}$ rt. angle, which is constant since π is a constant.

1-7. Relation between degrees and radians : From the previous article

$$\boxed{\pi \text{ radians} = 2 \text{ rt. angles} = 180^\circ} \quad \ldots\ldots\ldots(2)$$

$$\therefore \; 1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

$$= 57{\cdot}29577 \text{ degrees.}$$

$$= 57^\circ 17' 44{\cdot}8'' \text{ (approx.)}$$

and, $$1 \text{ degree} = \frac{\pi}{180} \text{ radians}$$

$$= {\cdot}0174533 \text{ radian (approx.)}$$

N. B. The formulae (2) connecting the sexagesimal and radian measures of an angle enable us to pass readily from one system to the other and is a very useful result.

While expressing the angles in circular measure, it is customary to drop the word 'radian', so that an angle $\theta = \pi$ means that the angle θ is π radians.

1-8. In trigonometrical calculations, the students will frequently require to convert the angles from one system of measurement to another and for the ready conversion we give below a table for certain important angles.

15°	=	**$\pi/12$ (radian)**
30°	=	**$\pi/6$,,**
45°	=	**$\pi/4$,,**
60°	=	**$\pi/3$,,**
75°	=	**$5\pi/12$,,**
90°	=	**$\pi/2$,,**
120°	=	**$2\pi/3$,,**
180°	=	**π ,,**
360°	=	**2π ,,**

4 **1-9. Length of an arc of a circle :** *The circular (or radian) measure of any angle at the centre of a circle is the ratio of the subtending arc to the radius.*

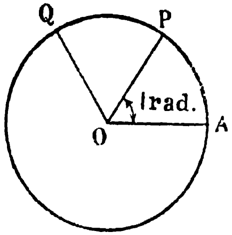

Fig. 7

Let AOQ be any angle at the centre of a circle, and $\angle AOP$ a radian ; so arc AP=radius.

Now, the radian measure of $\angle AOQ$

$$=\frac{\angle AOQ}{\angle AOP}$$

$$=\frac{\text{arc } AQ}{\text{arc } AP}$$

$$=\frac{\text{arc } AQ}{\text{radius}}.$$

Thus the radian measure of $\angle AOQ=\dfrac{\text{subtending arc}}{\text{radius}}$.

If l be the length of an arc which subtends an angle of θ radian at the centre of a circle of radius r,

$$\theta=\frac{l}{r}.$$

Or, $\boxed{\textbf{arc length}=\theta . r}$ ... ... (3)

Note : There is another system of measuring an angle : it is called the 'centesimal method' in which a right angle is divided into 100 equal parts called 'grades', a grade into 100 equal parts called 'minutes', a minute into 100 equal parts called 'seconds'. This system was proposed during the French revolution. But curiously enough, it has never been adopted even in France. Thus

1 rt. angle = 100g (grades)
1g = 100 (minutes)
1′ = 100″ (seconds)

1-10. Illustrated examples

Ex. 1. *Express in the sexagesimal system*

(i) One-twelfth of a right angle.

(ii) One-sixteenth of two right angles.

(i) We have, 1 rt. angle $=90°$

$\therefore \frac{1}{12}$ rt. angle $=90°/12$

$=7°30'$.

(ii) 2 rt. angles $=180°$

$\therefore \frac{1}{16}\times 2$ rt. angles $=180°/16=11°15'$.

Ex. 2. *Express in circular measure the angle 45° 15′ 27″.*

$$45^\circ\ 15'\ 27'' = 45{\cdot}2575 \text{ degrees.}$$

$\therefore$ the angle contains $\frac{\pi}{180} \times 45{\cdot}2575$ radians.

Ex. 3. *Express in sexagesimal measure the angle whose radian measure is $\frac{3}{5}$.*

Let D be the no. of degrees contained in the angle.

$$\therefore \quad \frac{D}{180} = \frac{3}{5\pi} \quad \therefore \quad D = \frac{3\times180}{5\pi} = \frac{3\times180\times7}{5\times22} \text{ degrees}$$

$$= 34^\circ 21' 49'' \quad (\pi = 22/7).$$

Ex. 4. *The angles of a triangle are in the ratio 1 : 2 : 3 ; express them in degrees.*

The angles are in A.P.

Let α, 2α, 3α be the angles.

$\therefore \quad \alpha + 2\alpha + 3\alpha = 180^\circ$

$\therefore \quad \alpha = 30^\circ$

$\therefore$ the other angles are 60°, 90°.

Ex. 5. *An arc 17 yds. 1 ft. 3 inches subtends at the centre of a circle an angle of 1·9 radian. Find the length of the radius in inches.*

We have, $\dfrac{\text{arc}}{\text{radius}} = \text{angle (in radian)}$

$$\therefore \quad \frac{\{(17\times3)+1\}12+3 \text{ inches}}{r \text{ (inches)}} = 1{\cdot}9$$

$$\text{radius} = \frac{52\times12+3}{1{\cdot}9} = \frac{624+3}{1{\cdot}9} = \frac{627}{1{\cdot}9} = 330 \text{ inches.}$$

Ex. 6. *At what distance does a tower $5\frac{1}{2}$ ft. in height subtend an angle 12′ at the eye ?*

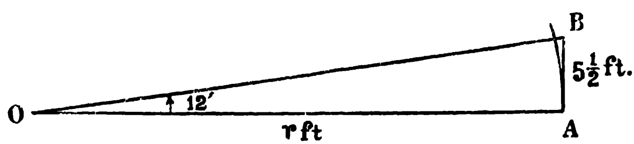

Fig. 8

Let AB be the tower ; O the observer ; so $\angle AOB = 12'$.

Since $\angle AOB$ is very small, AB may be taken as an arc of a circle with centre O.

$$\therefore \quad \frac{12}{60\times 60}\times\frac{\pi}{180}=\frac{5\frac{1}{2}}{r} \text{ where } r \text{ ft.} = OA.$$

$$\therefore \quad r=\frac{11}{2}\times\frac{180\times 60\times 60}{12\times\pi} \text{ ft.}=94500 \text{ ft.}=17\cdot 89 \text{ miles.}$$

Examples 1

1. Express in sexagesimal system
$3\pi/4$, $5\pi/4$, $\pi/54$, $1\frac{1}{2}$, $\cdot 02$, $1\cdot 309$.

2. Express in radian measure
$57^\circ 30'$, $14^\circ 24'$, $18^\circ 33' 45''$, $45^\circ 13' 30''$.

3. Find the no. of radians in each exterior angle of (i) a regular pentagon (ii) a regular decagon.

4. Express in radians the interior angle of a regular polygon of n sides and of a regular heptagon.

5. One angle of a triangle is 45° and another $5\pi/8$ radians; express the third both in sexagesimal and radian measure.

6. The number of degrees in an angle exceeds 14 times the number of radians in it by 51. Find the sexagesimal measure of the angle.

7. The difference of two angles is 10° and the circular measure of their sum is 2. Find the circular measure of each angle.

8. The sum of two angles is 3 radians and their difference is 10 degrees. Find each angle in degrees (assume $43\pi=135$).

9. The three angles of a triangle are in A. P. and the greatest is double the smallest. Find each angle in degrees.

10. The flywheel of an engine makes 40 revolutions a second; in what time will it turn through 9 radians ?

11. Find the number of radians in the angles of a triangle which are in arithmetical progression, the least angle being 36°.

12. The angles of a triangle are in A. P. and the number of degrees in the least is to the number of radians in the greatest as $60 : \pi$. Find the angles in radians.

13. The angles of a triangle are in A. P. The number of degrees in the greatest is to the number of radians in the sum of the other two as $450 : 11$. Find the angles in degrees.

14. In each of two triangles the angles are in G. P.; the least angle of one of them is three times the least angle in the other, and the sum of the greatest angles is 240°. Find the circular measure of the angles.

15. Find the number of radians in the complement of $3\pi/8$.

16. The angles of a polygon are in A. P. The least angle is $2\pi/3$ radians and the common difference is 5°. Find the number of sides (The polygon has no reflex angle).

17. Find the angle subtended by an arc 7·5 ft. at the centre of a circle whose radius is 5 yards.

18. An arc 20 yds. 2 ft. 6 inches subtends an angle of 1·5 radian at the centre of a circle. Find the length of the radius.

19. What is the height of a tower that subtends an angle $1'$ at the eye of an observer at a distance of 1 mile from the tower?

20. At what distance does a tower 10 ft. in height subtend an angle of $10'$ at the eye?

21. In running a race at a uniform speed on a circular track, a man in each minute traverses an arc of a circle which subtends $2\frac{6}{7}$ radians at the centre of the track. If each lap is 792 yds., how long does he take to run a mile?

22. Find the radius of a globe such that the distance measured along its surface between two places on the same meridian whose latitudes differ by $1°10'$ may be 1 inch ($\pi = 22/7$).

23. A horse is tethered to a stake by a rope of 27 ft. long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far it would have gone when the rope has traced out an an angle of 70°.

24. The angular diameter of the sun on two particular days in the year is found to be $31'$ and $32'$ respectively. Compare the earth-sun distances on these two days.

2

The trigonometrical ratios

2-1. We shall now define trigonometrical ratios or the circular functions. These are of fundamental importance in the subsequent development of trigonometry.

Let XOP be any acute angle. From any point P on OP,

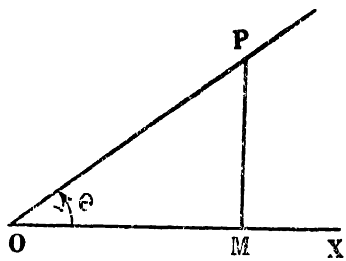

Fig. 9

one of the boundary lines, draw PM perp. to OX ; so $\triangle$POM is a rt. angled $\triangle$.

If θ be the measure of the angle, the trigonometrical ratios are defined as follows :

sine of the angle θ, written sin θ, is $\dfrac{MP}{OP}$

cosine of the angle θ, written cos θ, is $\dfrac{OM}{OP}$

tangent of the angle θ, written tan θ, is $\dfrac{MP}{OM}$

Thus

$$\sin\theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}}$$

$$\cos.\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta}$$

There are three other trigonometrical ratios which are the reciprocals of sine, cosine and tangent. They are respectively called *cosecant*, *secant* and *cotangent*. For brevity, they are written as *cosec θ*, *sec θ* and *cot θ*. Thus,

$$\textbf{cosec } \theta = \frac{\text{hypotenuse}}{\text{side opp. } \theta} = \frac{OP}{MP}$$

$$\textbf{sec } \theta = \frac{\text{hypotenuse}}{\text{side adj. to } \theta} = \frac{OP}{OM}$$

$$\textbf{cot } \theta = \frac{\text{side adj. to } \theta}{\text{side opp. } \theta} = \frac{OM}{MP}$$

In defining the trigonometrical ratios, we confined ourselves to *acute* angles. We shall show presently how the definitions may be extended to angles of any magnitude.

Note: (i) In addition to the above six trigonometrical ratios, two others are sometimes used. They are termed 'versed sin θ' (written, vers θ) and 'coversed sine θ' (written coverse θ) and are defined as:

$$\text{vers } \theta = 1 - \cos \theta$$
$$\text{coverse } \theta = 1 - \sin \theta$$

(ii) Trigonometrical ratios on an angle are numerical quantities, being the ratio of two lengths. All algebrical operations may, therefore, be performed upon them. They must never be regarded as lengths.

(iii) Before continuing with the study of trigonometry, it is important that the students should become thoroughly conversant with these definitions.

2-2. Signs to the trigonometrical ratios. Trigonometrical ratios of any angle

Let XX′ and YY′ be two straight lines intersecting at right

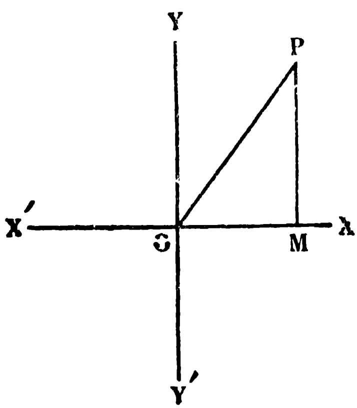

Fig. 10.

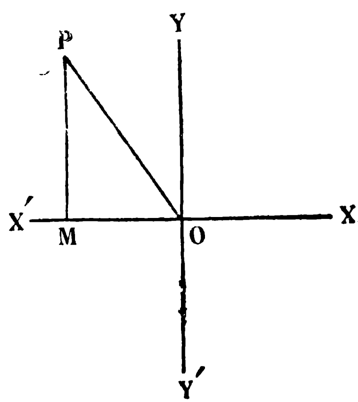

Fig. 11

angles in O. They divide the plane into four *quadrants* XOY,

YOX′, X′OY′, Y′OX, named *first*, *second*, *third*, *fourth* in order.

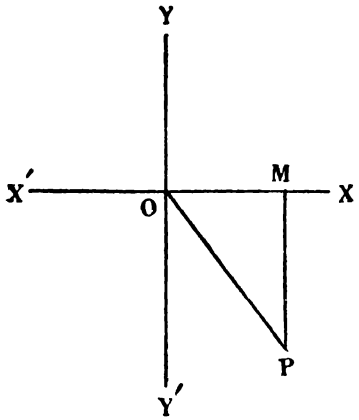

Fig. 12

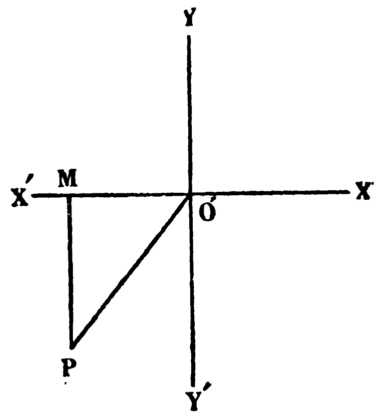

Fig. 13

The distances measured along or parallel to OX or OY are conventionally taken as *positive* and those measured along or parallel to OX′ or OY′ *negative.*

Let now the generating line starting from OX revolve in either direction, clockwise or counter-clockwise and assume the position OP by tracing out an angle θ.

From a point P on the final position of the generating line draw PM perp. to XX′. Then, in the rt. angled $\triangle$ OPM, in whatever quadrant OP happens to lie, the ratios of the last article, are called the trigonometrical ratios of θ. That is, with due regard paid to the sign convention,

$$\sin\theta=\frac{PM}{OP}\,; \quad \cos\theta=\frac{OM}{OP} \text{ etc.}$$

The line OP, which only fixes the boundary of the angle, is considered to be always positive.

These considerations will immediately lead to the following conclusions as regards the sign of the trigonometrical ratios.

In the *first quadrant*, OP, PM, OM are all positive and therefore, *all the trigonometrical ratios are positive.*

In the *second quadrant*, OP and PM are positive, OM is negative ; so *sin θ* and its reciprocal *cosec θ* are *positive* and all others are *negative.*

In the *third quardrant,* OP is positive, OM and PM are negative ; hence *tan θ* and its reciprocal *cot θ* are *positive* and all others are *negative.*

In the *fourth quadrant,* OP and OM are positive, PM is negative ; so *cos θ* and its reciprocal *sec θ* are *positive* and others are *negative.*

An inspection of the diagram below will help remember the signs of trigonometrical ratios.

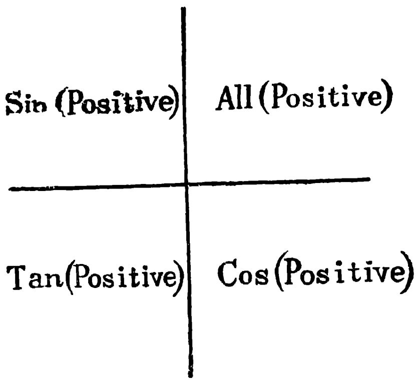

Fig. 14

2-3. Coterminal angles

When any angle is increased or diminished by any multiple of 2π (radians), the generating line takes up the same position after one or more complete revolutions. Such angles which have the same boundary line are called *coterminal angles.* All the angles coterminal with θ is expressed by the general form $2n\pi+\theta$ where n is any integer.

It is obvious that *all coterminal angles have the same trigonometrical ratios* ; e. g., $\sin(2n\pi+\theta)=\sin\theta$.

2·4. For a given angle, the trigonometrical ratios are unique

Let XOY be any acute angle of measure θ. Take any two

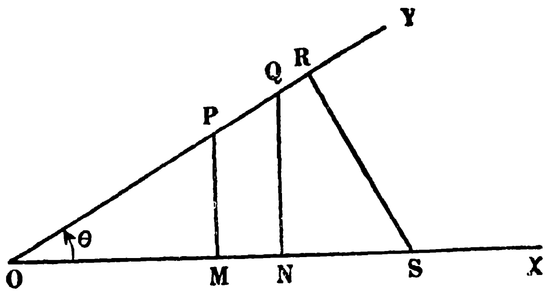

Fig. 15

points P, Q on OY and drop perps. PM and QN on OX. Let R be another point on OY and draw RS perp. to OY.

$\therefore$ $\sin\theta=\dfrac{PM}{OP}$, from the rt. angled $\triangle$ POM

also, $\sin\theta=\dfrac{QN}{OQ}$, from the rt. angled $\triangle$ QON

and $\sin\theta=\dfrac{RS}{OS}$, from the rt. angled $\triangle$ SOR

But the $\triangle^s$ POM, QON, SOR are all equiangular,

$$\therefore \quad \frac{PM}{OP}=\frac{QN}{OQ}=\frac{RS}{OS}.$$

Thus, the sine of the angle θ is the same whether we take it from $\triangle$ POM or from $\triangle$ QON or from $\triangle$ SOR. This is true for all other ratios and for angles of all magnitude. For a given angle, therefore, the trigonometrical ratios are unique.

2-5. Relations between trigonometrical ratios of an angle

From the very definition it follows that

$$\boxed{\begin{aligned} \operatorname{cosec}\theta &= \frac{1}{\sin\theta} \\ \sec\theta &= \frac{1}{\cos\theta} \\ \cot\theta &= \frac{1}{\tan\theta} \end{aligned}} \quad \ldots\ldots\ldots(4)$$

Also, since, with reference to the Fig. 1

$$\sin\theta=\frac{PM}{OP} \text{ and } \cos\theta=\frac{OM}{OP},$$

we have, $\dfrac{Sin\,\theta}{Cos\,\theta}=\dfrac{PM}{OP}\div\dfrac{OM}{OP}=\dfrac{PM}{OM}=tan\,\theta.$

Thus,

$$\boxed{\begin{array}{c}\tan\theta=\dfrac{\sin\theta}{\cos\theta}\\[2ex] \cot\theta=\dfrac{\cos\theta}{\tan\theta}\end{array}} \quad \ldots\ldots\ldots(5)$$

2-6. Other connecting relations

Since MOP is a rt. angled $\triangle$, rt. angled at **M**, we have

$$OP^2=PM^2+OM^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A)$$

$$\therefore\quad 1=\left(\frac{PM}{OP}\right)^2+\left(\frac{OM}{OP}\right)^2$$

or $\quad 1=(\sin\theta)^2+(\cos\theta)^2$

or, $\quad 1=\sin^2\theta+\cos^2\theta$ as is usually written.

From (A), it also follows that

$$\left(\frac{OP}{PM}\right)^2=1+\left(\frac{OM}{PM}\right)^2$$

or, $\operatorname{cosec}^2\theta=1+\cot^2\theta,$

and, $\quad \left(\dfrac{OP}{OM}\right)^2=1+\left(\dfrac{PM}{OM}\right)^2,$

or, $\sec^2\theta=1+\tan^2\theta.$

Collecting the relations,

$$\boxed{\begin{array}{ll}\sin^2\theta+\cos^2\theta=1 & \\ 1+\tan^2\theta & =\sec^2\theta\\ 1+\cot^2\theta & =\operatorname{cosec}^2\theta.\end{array}}$$

Cor :

$$\sin^2\theta=1-\cos^2\theta$$
$$\cos^2\theta=1-\sin^2\theta$$
$$\tan^2\theta=\sec^2\theta-1$$
$$\cot^2\theta=\operatorname{cosec}^2\theta-1$$

Note : If we replace θ by $\theta/2$, say, then the formulæ would be $\sin^2\theta/2+\cos^2\theta/2=1$ etc.

With the help of the formulæ of Art 2-5 and 2-6, we can express any trigonometrical ratio in terms of any other for a given angle.

2-7. Restrictions in the value of trigonometrical ratios

In a rt. angled $\triangle$, the hypotenuse is the greatest side; hence, from the definitions of the trigonometrical ratios, it follows that

the sine and cosine of an angle cannot be >1 numerically. the cosecant and secant of an angle can never be <1 numerically and the tangent and cotangent may have any numerical value.

2-8. Illustrated examples

Ex. 1. *Express all the trigonometrical functions in terms of sine.*

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\therefore \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}$$

$$\therefore \cot\theta = \frac{1}{\tan\theta} = \frac{\pm\sqrt{1-\sin^2\theta}}{\sin\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\pm\sqrt{1-\sin^2\theta}}$$

Ex. 2. *Express all the functions in terms of the tangent.*

$$\sec^2\theta = 1 + \tan^2\theta.$$

$$\therefore \sec\theta = \pm\sqrt{1+\tan^2\theta}.$$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\pm\sqrt{1+\tan^2\theta}}$$

$$\sin\theta = \tan\theta\cos\theta = \frac{\tan\theta}{\pm\sqrt{1+\tan^2\theta}}$$

$$\operatorname{cosec}\theta = \frac{\pm\sqrt{1+\tan^2\theta}}{\tan\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Note : It is always possible to construct a rt. angled $\triangle$, if two sides are given. By this method we can readily get all trigonometrical functions, when one is given.

Ex. 3. *If* $\tan\alpha=\frac{1}{2}$, *find* $\sec\alpha$ *and* $\operatorname{cosec}\alpha$.

$\because$ $\tan\alpha=\frac{1}{2}$, we describe the rt. angled $\triangle$ ABC, with sides AC=1 unit, BC=2 units, so that $\angle ABC=\alpha$.

From geometry,

$$AB=\sqrt{1^2+2^2}=\sqrt{5}$$

$$\therefore\quad \sec\alpha=\frac{AB}{BC}=\frac{\sqrt{5}}{2},$$

$$\operatorname{cosec}\alpha=\frac{AB}{BC}=\sqrt{5}.$$

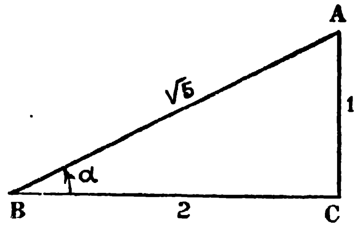

Fig. 16

Ex. 4. *Prove that* $\sin^2 A\tan A+\cos^2 A\cot A+2\sin A\cos A=\tan A+\cot A$.

$$\text{Left side}=\sin^2 A\,\frac{\sin A}{\cos A}+\cos^2 A\,\frac{\cos A}{\sin A}+2\sin A\cos A$$

$$=\frac{\sin^4 A+\cos^4 A+2\sin^2 A\cos^2 A}{\sin A\cos A}$$

$$=\frac{(\sin^2 A+\cos^2 A)^2}{\sin A\cos A}$$

$$=\frac{1}{\sin A\cos A}$$

$$\text{Right side}=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}$$

$$=\frac{\sin^2 A+\cos^2 A}{\sin A\cos A}$$

$$=\frac{1}{\sin A\cos A}.$$

Ex. 5. *Show that* $\sqrt{\dfrac{1-\sin A}{1+\sin A}}=\sec A-\tan A$.

$$\text{Left side}=\sqrt{\frac{1-\sin A}{1+\sin A}\cdot\frac{1-\sin A}{1-\sin A}}\quad\left(\because\ \frac{1-\sin A}{1-\sin A}=1\right)$$

$$=\frac{1-\sin A}{\cos A}$$

$$=\frac{1}{\cos A}-\frac{\sin A}{\cos A}$$

$$=\sec A-\tan A.$$

Ex. 6. *If $\sin A+\sin^2 A=1$; prove that $\cos^2 A+\cos^4 A=1$*

$\because$ $\sin A+\sin^2 A=1$, we get by transposition,

$$1-\sin^2 A=\sin A$$

or, $\cos^2 A=\sin A$.

$$\therefore \cos^2 A+\cos^4 A=\sin A+(\cos^2 A)^2$$
$$=\sin A+(\sin A)^2$$
$$=\sin A+\sin^2 A$$
$$=1.$$

Ex. 7. *If $\tan\theta+\sec\theta=x$, show that $\sin\theta=\dfrac{x^2-1}{x^2+1}$*

$\because$ $\tan\theta+\sec\theta=x$, we have

$$\frac{1+\sin\theta}{\cos\theta}=x.$$

$$\therefore \quad x^2=\frac{(1+\sin\theta)^2}{\cos^2\theta}.$$ By comp. and dividendo,

$$\frac{x^2-1}{x^2+1}=\frac{(1+\sin\theta)^2-\cos^2\theta}{(1+\sin\theta)^2+\cos^2\theta}=\frac{2\sin\theta+2\sin^2\theta}{2+2\sin\theta}$$

$$=\frac{2\sin\theta\,(1+\sin\theta)}{2(1+\sin\theta)}=\sin\theta.$$

Ex. 8. *If $\tan\theta=\dfrac{a}{b}$, show that $\dfrac{a\sin\theta-b\cos\theta}{a\sin\theta+b\cos\theta}=\dfrac{a^2-b^2}{a^2+b^2}$*

Given, $\tan\theta=\dfrac{a}{b}$

or, $$\frac{a\sin\theta}{b\cos\theta}=\frac{a^2}{b^2}$$

$$\therefore \quad \frac{a\sin\theta-b\cos\theta}{a\sin\theta+b\cos\theta}=\frac{a^2-b^2}{a^2+b^2}$$ (By comp. & div.)

Ex. 9. *If x and y are two unequal quantities, show that $\sin^2\theta=\dfrac{(x+y)^2}{4xy}$ is impossible.*

$$\sin^2\theta=\frac{(x+y)^2}{4xy}=\frac{(x+y)^2}{(x+y)^2-(x-y)^2}>1$$

Hence the equation is impossible.

Ex. 10. *If $\cos^2 A-\sin^2 A=\tan^2 B$, then*
$\cos^2 B-\sin^2 B=\tan^2 A$.

$\because\quad \cos^2 A-\sin^2 A=\tan^2 B$

$\therefore\quad (1-\sin^2 A)-\sin^2 A=\tan^2 B$

$\therefore\quad 1-2\sin^2 A=\tan^2 B$

$\therefore\quad 2\sin^2 A=1-\tan^2 B$

Again, $\because\quad \cos^2 A-\sin^2 A=\tan^2 B$

$\therefore\quad \cos^2 A-(1-\cos^2 A)=\tan^2 B$

$\therefore\quad 2\cos^2 A-1=\tan^2 B,\quad \therefore\ 2\cos^2 A=1+\tan^2 B$

$$\tan^2 A=\frac{2\sin^2 A}{2\cos^2 A}=\frac{1-\tan^2 B}{1+\tan^2 B}=\frac{1-\dfrac{\sin^2 B}{\cos^2 B}}{1+\dfrac{\sin^2 B}{\cos^2 B}}=\frac{\cos^2 B-\sin^2 B}{\cos^2 B+\sin^2 B}$$

$$=\cos^2 B-\sin^2 B.$$

Examples 2

Prove that

1. $\sin^2 A\cot^2 A+\cos^2 A\tan^2 A=1$
2. $\sin A\sqrt{1-\sin^2 A}=\sin A\cos A$
3. $(1+\tan^2 A)\cos^2 A=1$
4. $\sin A\,(1+\tan A)+\cos A\,(1+\cot A)=\sec A+\operatorname{cosec} A$
5. $\sec^4\theta-\sec^2\theta=\tan^2\theta+\tan^4\theta$
6. $\dfrac{\sin A+\cos A}{\sec A+\operatorname{cosec} A}=\sin A\cos A$
7. $\tan^2 A+\cot^2 A=\dfrac{1-2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$
8. $\dfrac{(\operatorname{cosec}\theta+\sec\theta)^2}{\sec^2\theta+\operatorname{cosec}^2\theta}=1+2\sin\theta\cos\theta$
9. $(\tan\theta+\sec\theta)^2=\dfrac{1+\sin\theta}{1-\sin\theta}$ (C. U. 1934)
10. $(\cot\theta+\operatorname{cosec}\theta)^2=\dfrac{1+\cos\theta}{1-\cos\theta}$
11. $\dfrac{2\sin\theta\cos\theta-\cos\theta}{1-\sin\theta+\sin^2\theta-\cos^2\theta}=\cot\theta$
12. $\tan^2\alpha+\sec^2\beta=\sec^2\alpha+\tan^2\beta$

13. $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\left(\dfrac{1-\tan\theta}{1-\cot\theta}\right)^2$

14. $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\dfrac{1+\sin\theta}{\cos\theta}=\sec\theta+\tan\theta$

15. $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}=\operatorname{cosec}\theta+\cot\theta$

16. $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}-\sec\theta=\sec\theta-\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

17. $\dfrac{\operatorname{cosec}A+\cot A}{\operatorname{cosec}A-\cot A}=\dfrac{\sin^2 A}{(1-\cos A)^2}$

18. If $\cos A=\frac{5}{13}$, find (geometrically) cosec A, cot A

19. If $\sin A=\dfrac{m}{n}$, prove that $\sqrt{n^2-m^2}\tan A=m$

20. If $\cot\theta=\dfrac{p}{q}$, what is the value of $\dfrac{p\cos\theta-q\sin\theta}{p\cos\theta+q\sin\theta}$?

21. If $\cos A=m$, $\cot A=n$, prove that $(1-m^2)(1+n^2)=1$

22. If $\tan A=\dfrac{2pq}{p^2-q^2}$, find cos A and sec A

23. If $\sec A=\dfrac{m^2+1}{2m}$ find cos A and cosec A

24. If $\sin\theta=\dfrac{m^2-n^2}{m^2+n^2}$, find $\cos\theta$ and $\tan\theta$

25. If $\tan\theta=\dfrac{m}{n}$, prove that $m^2\operatorname{cosec}^2\theta=m^2+n^2$

26. If $\dfrac{\sin\theta}{a+b}=\dfrac{\cos\theta}{a-b}$, find $\sin\theta$ and $\cos\theta$

27. If $\sin\theta-\cos\theta=p$ and $\sec\theta-\operatorname{cosec}\theta=q$, prove that $q(1-p^2)=2p$

28. If $\tan\theta+\sin\theta=m$ and $\tan\theta-\sin\theta=n$, prove that $m^2-n^2=4\sqrt{mn}$

29. If $\sin A-\cos A=0$, show that $\sec A=\pm\sqrt{2}$

30. If $a\cos\theta+b\sin\theta=1$ and $c\cos\theta+d\sin\theta=1$, show that $(a-c)^2+(b-d)^2=(bc-ad)^2$

31. Express $1-2\sin\theta\cos\theta$ as a perfect square

32. Express $\sin^2\theta\tan\theta+\cos^2\theta\cot\theta+2\sin\theta\cos\theta$ in terms of $\tan\theta$

33. If $\tan^2\theta=1-e^2$, show that $\sec\theta+\tan^3\theta\operatorname{cosec}\theta=(2-e^2)^{\frac{3}{2}}$

34. If $15 \sin^2 \theta + 2 \cos \theta = 7$, find $\tan \theta$

35. If $7 \sin^2 \theta + 3 \cos^2 \theta = 4$, show that $\tan \theta = \pm \frac{1}{\sqrt{3}}$

36. If $1 + \sin^2 \theta = 3 \sin \theta \cos \theta$, find $\cot \theta$

37. If $\cos \alpha + \sin \alpha = \sqrt{2} \cos \alpha$, prove that
$\cos \alpha - \sin \alpha = \sqrt{2} \sin \alpha$

38. If $\cos^2 \theta - \sin^2 \theta = \tan^2 \phi$, prove that
$\cos^2 \phi - \sin^2 \phi = \tan^2 \theta$

39. Find the value of $\sin \theta - \cos \theta$, when $\tan \theta = 2$

40. Show that $\cos \theta = x + \frac{1}{x}$ is impossible if x and y are two unequal real quantities

41. Show that $\sec^2 \theta = \frac{4xy}{(x+y)^2}$ is possible if $x = y$; what is the value of θ then ?

Prove that

42. $\sin \theta (1 + \tan \theta) + \cos \theta (1 + \cot \theta) = \sec \theta + \operatorname{cosec} \theta$

43. $\frac{1}{\operatorname{cosec} \theta - \cot \theta} - \frac{1}{\sin \theta} = \frac{1}{\sin \theta} - \frac{1}{\operatorname{cosec} \theta + \cot \theta}$

44. $\cos \theta + \frac{\tan \theta \cos \theta}{\sin \theta} + \frac{\sin^2 \theta}{\cos \theta} = 1 + \sec \theta$

45. $\frac{\tan \alpha}{(1+\tan^2 \alpha)^2} + \frac{\cot \alpha}{(1+\cot^2 \alpha)^2} = \sin \alpha \cos \alpha$

46. $\frac{\tan^2 \alpha + \cot^2 \alpha}{\tan^2 \alpha - \cot^2 \alpha} = \frac{\sin^4 \alpha + \cos^4 \alpha}{\sin^2 \alpha - \cos^2 \alpha}$

47. If $\tan^2 A = 1 + 2 \tan^2 B$, show that $\cos^2 B = 2 \cos^2 A$

48. If $x \sin^3 \alpha + y \cos^3 \alpha = \sin \alpha \cos \alpha$ and
$x \sin \alpha - y \cos \alpha = 0$, show that $x^2 + y^2 = 1$

49. Prove that $(2 - \cos^2 A)(1 + 2 \cot^2 A)$
$= (2 + \cot^2 A)(2 - \sin^2 A)$

3

The trigonometrical ratios of certain angles

3-1. Ratios of 45° or $\frac{\pi}{4}$

Let ABD be a rt. angled isosceles $\triangle$ rt. angled at D ; hence $\angle B = \angle A = 45°$.

Let each of the equal sides BD and AD, contain a units.

$\therefore \quad AB = \sqrt{BD^2 + AD^2} = \sqrt{a^2 + b^2}$
$= \sqrt{2}a.$

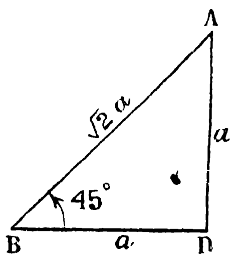

Fig. 17

$\therefore \quad \sin 45° = \frac{AD}{AB} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{BD}{AB} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$

$\tan 45° = \frac{AD}{BD} = \frac{a}{a} = 1.$

and so, cosec $45° = \sqrt{2}$, sec $45° = \sqrt{2}$, cot $45° = 1$

3-2. Ratios of 60° and 30° or $\frac{\pi}{3}$ and $\frac{\pi}{6}$

Let ABC be an equilateral $\triangle$; so each of its sides is equal to $2a$, say, and each of its angles is 60°.

Draw AD perp. to BC ; so AD bisects the $\angle BAC$ and the base BC.

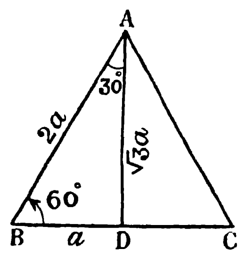

Fig. 18

$\therefore \quad \angle ABD = 60°$; $\angle BAD = 30°$,
$BD = a$ and $AD = \sqrt{AB^2 - BD^2}$
$= \sqrt{4a^2 - a^2} = \sqrt{3}a$

$$\therefore \quad \sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

Again, $$\sin 30^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

The corresponding other ratios are :

$\operatorname{cosec} 60^\circ = 2/\sqrt{3}$, $\sec 60^\circ = 2$, $\cot 60^\circ = 1/\sqrt{3}$

$\operatorname{cosec} 30^\circ = 2$, $\sec 30^\circ = 2/\sqrt{3}$, $\cot 30^\circ = \sqrt{3}$

3-3. Ratios of 90° (or $\pi/2$) and 0°

Let BOC be a quadrant of a circle. From any point P on the circumference draw PN perp. to OC. Then if $\angle PON = \theta$, $\sin \theta = \frac{PN}{OP}$, $\cos \theta = \frac{ON}{OP}$ etc.

If now P approaches B, the angle θ becomes greater and greater and so also PN ; ON becomes shorter, but OP remains constant in length. As P becomes coincident with B, the angle θ becomes 90°, ON vanishes and PN becomes equal to OP.

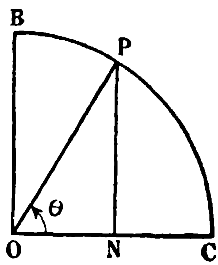

Fig. 19

$$\therefore \sin 90^\circ = \frac{PN}{OP} \text{ in the limit} = 1$$

$$\cos 90^\circ = \frac{ON}{OP} \text{ in the limit} = 0$$

$$\tan 90^\circ = \frac{PN}{ON} \text{ in the limit} = \frac{OP}{0}$$

$$= \infty \text{ (infinity)}$$

(∞ stands for an infinitely large quantity.)

So, $\operatorname{cosec} 90^\circ = 1$, $\sec 90^\circ = \frac{1}{0} = \infty$, $\cot 90^\circ = \frac{1}{\infty} = 0$.

If, however, P approaches C, the angle θ becomes smaller and smaller and consequently PN becomes shorter. In the limit when P coincides with C, θ becomes zero, PN vanishes and ON becomes equal to OP.

$$\therefore \quad \sin 0° = \frac{PN}{OP} \text{ in the limit} = 0$$

$$\cos 0° = \frac{ON}{OP} \text{ in the limit} = 1$$

$$\tan 0° = \frac{PN}{ON} \text{ in the limit} = \frac{0}{OP} = 0$$

So, cosec $0° = \infty$, sec $0° = 1$ cot $0° = \infty$.

3-4. The above values of the trigonometrical ratios are frequently required. The students should get them memorised. We draw up a table as follows.

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

The other ratios, namely cosec, sec, and cot are reciprocal to these in order.

3-5. Illustrated examples

Ex. 1. *Find the value of 2 sin 30° cos 30° cot 60°*

$$\text{The value} = 2\,\frac{1}{2}.\ \frac{\sqrt{3}}{2}.\ \frac{1}{\sqrt{3}}$$

$$= \frac{1}{2}.$$

Ex. 2. *Verify that* $\dfrac{2 \tan\frac{\pi}{6}}{1+\tan^2\frac{\pi}{3}} = \sin\frac{\pi}{3}$

$$\text{Left side} = \frac{2.\ \frac{1}{\sqrt{3}}}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\frac{4}{3}} = \frac{\sqrt{}}{2}$$

$$\text{Right side} = \frac{\sqrt{3}}{2}. \quad \text{Hence.}$$

Ex. 3. *Solve for θ, where θ is a positive acute angle, from the equation $4\cos^2\theta-\sqrt{2}\cos\theta-1=0$.*

Treating it as a quadratic in $\cos\theta$, we get

$$\cos\theta=\frac{1}{\sqrt{2}}\quad\text{or}-\frac{1}{2\sqrt{2}};$$

since θ is acute, $\cos\theta$ cannot be negative.

$$\therefore\quad \theta=45^\circ,\ \text{from}\ \cos\theta=\frac{1}{\sqrt{2}}$$

Examples 3

Find the values of

1. $\cos 30^\circ\cos 60^\circ+\sin 30^\circ\sin 60^\circ$
2. $2\operatorname{cosec}^2 45^\circ+\cot 60^\circ\tan 30^\circ$
3. $\tan^2 45^\circ\sin 60^\circ\tan 30^\circ\tan^2 60^\circ$
4. $4\sin^2 30^\circ\cos^2 60^\circ\cot^3 45^\circ\sec^4 45^\circ$
5. $2\cot 45^\circ+\cos^3 60^\circ-2\sin^4 60^\circ+\frac{3}{4}\tan^2 30^\circ$

Verify that

6. $\tan 45^\circ=\tan 30^\circ\tan 60^\circ$
7. $\dfrac{1-\tan^2 30^\circ}{1+\tan^2 30^\circ}=\cos 60^\circ$
8. $\dfrac{1+\sin 60^\circ}{\sin 30^\circ}=\dfrac{1+\tan 30^\circ}{1-\tan 30^\circ}$
9. $\tan(60^\circ+30^\circ)=\dfrac{\tan 60^\circ+\tan 30^\circ}{1-\tan 60^\circ\tan 30^\circ}$
10. $\sin 3\alpha=3\sin\alpha\cos\alpha$ when $\alpha=\dfrac{\pi}{3}$

Evaluate x from the following

11. $x\sin 30^\circ\cos^2 45^\circ=\dfrac{\cot^2 30^\circ\sec 60^\circ\tan 45^\circ}{\operatorname{cosec}^2 45^\circ\operatorname{cosec} 30^\circ}$
12. $x\cos 60^\circ\sin^2 45^\circ=\dfrac{\tan^2 60^\circ\sec 60^\circ\cot 45^\circ}{\sec^2 45^\circ\sec^2 60^\circ}$
13. $x\sin 60^\circ\cos^2 30^\circ=\dfrac{\tan^2 45^\circ\sec 60^\circ}{\cot^3 45^\circ}$

Solve for θ, from the following equations, θ being a positive acute angle

14. (i) $2\sin^2\theta=3\cos\theta$
    (ii) $2\cos\theta=\cot\theta$ (iii) $6\cos^2\theta-1=\cos\theta$
    (iv) $3\sqrt{3}\sec\theta\tan\theta+1=3\sec\theta+\sqrt{3}\tan\theta$
    (v) $\sec\theta+2\sqrt{2}\cos\theta=2+\sqrt{2}$
15. *If θ and ϕ are positive acute angles and* $\tan(\theta+\phi)=\sqrt{3}$, $\tan(\theta-\phi)=1$, *find θ* and ϕ

4

The trigonometrical ratios of associated angles

4-1. Ratios of the angle $(-\theta)$, θ having any magnitude

Let the $\angle$XOP be θ and $\angle$XOQ, traced counterclockwisely, be $-\theta$. From any point P on the boundary line draw PM perp. to OX [or OX$'$ as in (ii) and (iii)] and produce it to meet OQ at Q.

Since $\angle XOP = \angle XOQ$, in magnitude (not in sign), the $\angle^s$ POM add QOM are equal ; hence $\triangle^s$ POM QOM are congruent.

$$\therefore \quad QM = -PM,$$
$$OQ = OP,$$

the negative sign is due to the sign convention.

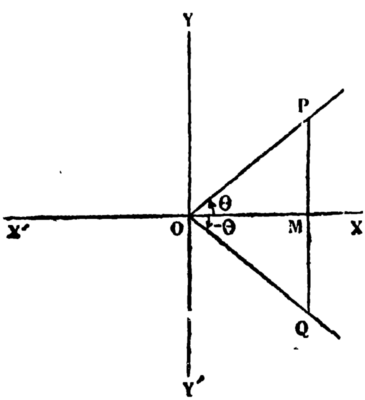

Fig. 20 (i)

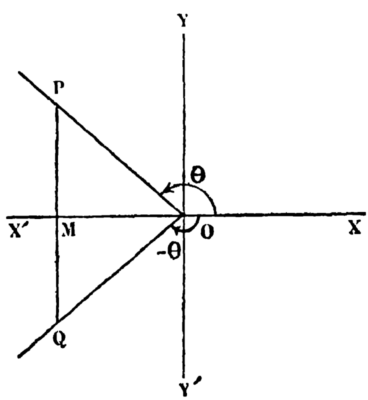

Fig. 20 (ii)

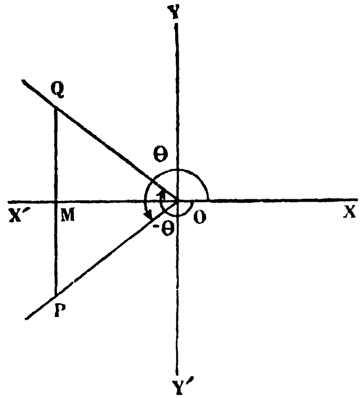

Fig. 20 (iii)

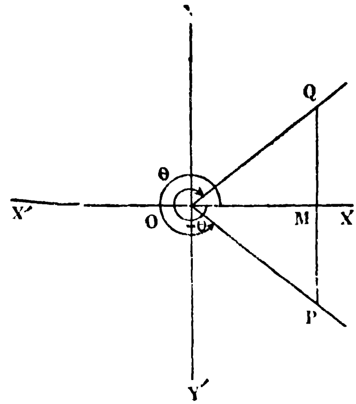

Fig. 20 (iv)

$$\therefore \quad \sin\ (-\theta)=\frac{QM}{OQ}=\frac{-PM}{OP}=-\sin\theta.$$

$$\cos\ (-\theta)=\frac{OM}{OQ}=\frac{OM}{OP}=\cos\theta.$$

$$\tan\ (-\theta)=\frac{QM}{OM}=\frac{-PM}{OM}=-\tan\theta.$$

So, $\operatorname{cosec}(-\theta)=-\operatorname{cosec}\theta$, $\sec(-\theta)=\sec\theta$,
$\cot(-\theta)=-\cot\theta$.

Note. We may change the sign of any angle without changing the value of its cosine and secant.

4-2. Ratios of $(90°-\theta)$

Let the $\angle XOP$ be θ and let a revoling line, starting from OX, trace out counterclockwisely the angle $XOY=90°$ and then rotate clockwisely by θ to take up finally the position OQ. So $\angle XOQ=90°-\theta$.

Take two points P, Q on OP and OQ respectively, such that $OP=OQ$. Draw PM, QN perps. to OX (or OX').

Now, since, $\angle XOP=\angle YOQ$, in magnitude (not in sign)

$\therefore \quad \angle POM=\angle OQN$

Also, since, $\quad OP=OQ$,

$\therefore \quad \triangle POM=\triangle OQN$

$\therefore$ $QN=OM$, $ON=PM$, $OQ=OP$, considering the signs as well.

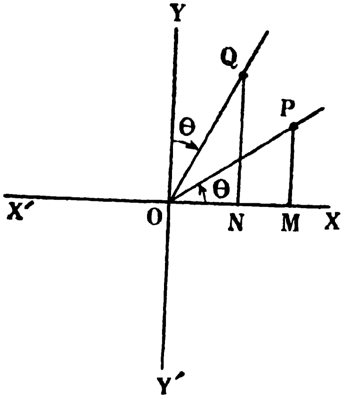

Fig. 21 (i)

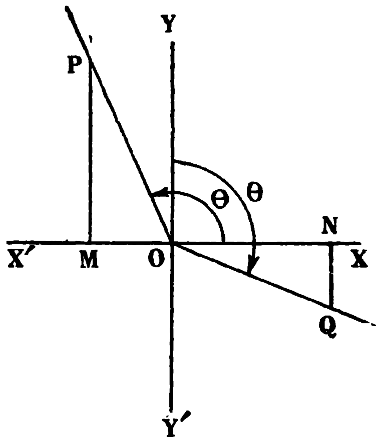

Fig. 21 (ii)

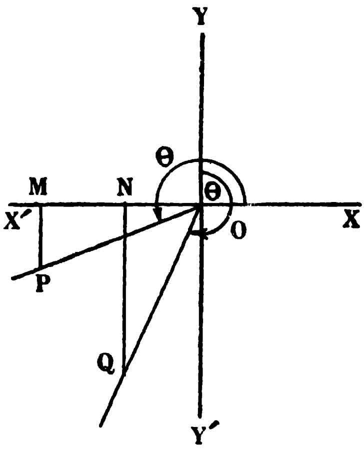

Fig. 21 (iii)

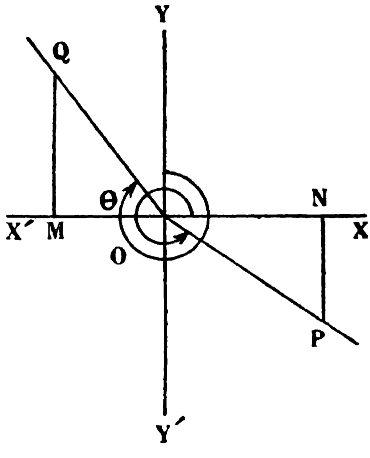

Fig. 21 (iv)

$$\therefore \quad \sin\ (90^\circ-\theta)=\sin\ XOQ=\frac{QN}{OQ}=\frac{OM}{OP}=\cos\theta.$$

$$\cos\ (90^\circ-\theta)=\frac{ON}{OQ}=\frac{PM}{OP}=\sin\theta.$$

$$\tan\ (90^\circ-\theta)=\frac{QN}{ON}=\frac{OM}{PM}=\cot\theta.$$

So, $\operatorname{cosec}(90^\circ-\theta)=\sec\theta$, $\sec(90^\circ-\theta)=\operatorname{cosec}\theta$,

$$\cot(90^\circ-\theta)=\tan\theta.$$

Note ; The angle $(90^\circ-\theta)$ is called the **complement** of θ. Each trigonometrical ratio of an angle is, thus, equal to the corresponding co-ratio of its complement. This may be verified from the table of values of trigonometrical ratios of some important angles given in the chapter.

4-3. Ratios of $(90°+\theta)$

Let the angle XOP generated by the revolving line OP

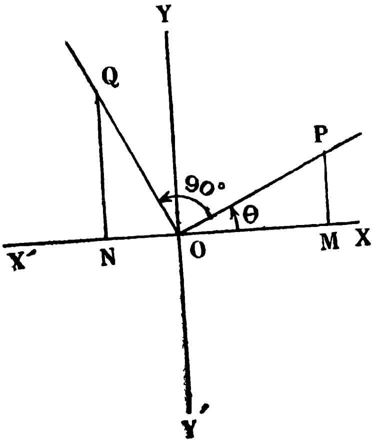

Fig. 22 (i)

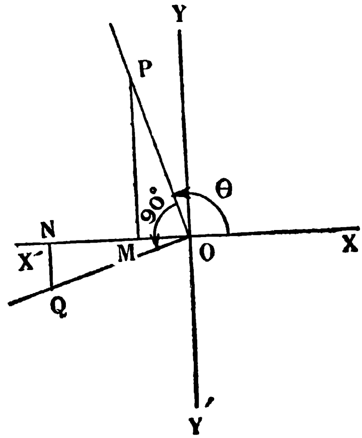

Fig. 22 (ii)

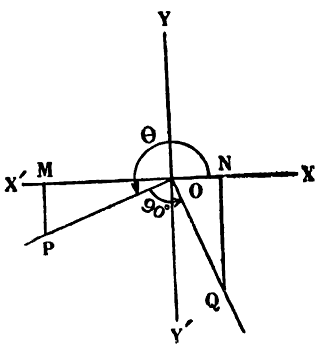

Fig. 22 (iii)

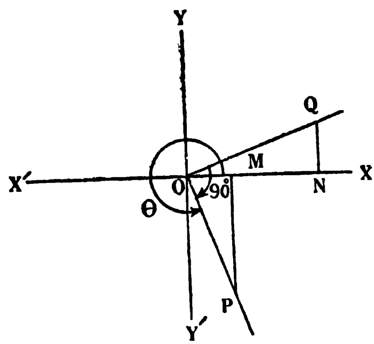

Fig. 22 (iv)

counter-clockwisely be θ and let it further trace the angle POQ=90° by rotating in the same direction.

$$\text{So,} \quad \angle XOQ = 90° + \theta$$

Take two points P, Q on OP and OQ respectively, such that OP=OQ. Draw PM and QN perps. to OX (or OX').

Since OQ is perp. to OP, the $\angle POM = \angle OQN$, both being complementary to $\angle QON$.

So, the $\triangle$s OPM, OQN are congruent ; hence, considering the signs as well,

$$QN = OM, \quad ON = -PM, \quad OQ = OP.$$

$$\therefore \quad \sin(90^\circ+\theta) = \sin XOQ = \frac{QN}{OQ} = \frac{OM}{OP} = \cos\theta.$$

$$\cos(90^\circ+\theta) = \frac{ON}{OQ} = \frac{-PM}{OP} = -\sin\theta.$$

$$\tan(90^\circ+\theta) = \frac{QN}{ON} = \frac{OM}{-PM} = -\cot\theta.$$

So, $\operatorname{cosec}(90^\circ+\theta) = \sec\theta$, $\sec(90^\circ+\theta) = -\operatorname{cosec}\theta$,

$\cot(90^\circ+\theta) = -\tan\theta$.

4-4. Ratios of $(180^\circ-\theta)$

Let the $\angle XOP$ be θ and let a revolving line, starting from

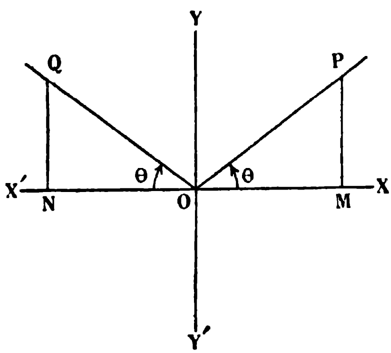

Fig. 23 (i)

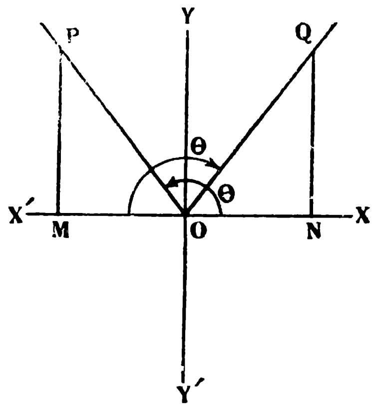

Fig. 23 (i)

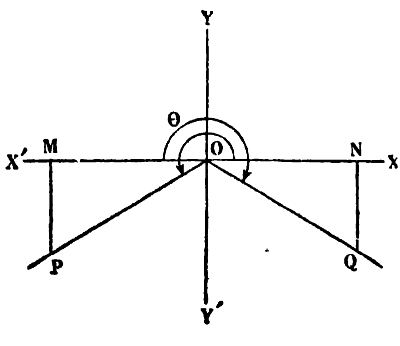

Fig. 23 (iii)

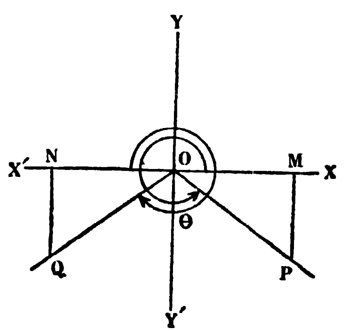

Fig. 23 (iv)

OX, trace out counter-clockwisely the angle $XOX' = 180^\circ$ and

then rotate clockwisely by θ to take up finally the position OQ. So, $\angle XOQ = 180° - \theta$.

Take OP=OQ and draw PM and QN perps. to OX (or OX'). Since, the $\angle POM = \angle QON$ in magnitude, and OP=OQ, the rt. angled $\triangle$s POM and QON are congruent. So, considering the signs as well,

$$QN = PM,\ ON = -OM,\ OQ = OP$$

$$\therefore\ \sin(180° - \theta) = \sin XOQ = \frac{QN}{OQ} = \sin\theta$$

$$\cos(180° - \theta) = \frac{ON}{OQ} = \frac{-OM}{OP} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{QN}{ON} = \frac{OP}{-OM} = -\tan\theta$$

So, $\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta$, $\sec(180° - \theta) = -\sec\theta$,
$\cot(180° - \theta) = -\cot\theta$.

Note: The angle $180° - \theta$ is called the **supplement** of θ. So sines of supplementary angles are equal but cosines and tangents of supplementary angles are equal in magnitude but opposite in sign.

4-5. Ratios of $(180° + \theta)$

Let the $\angle XOP$ traced by the revolving line OP be θ and let

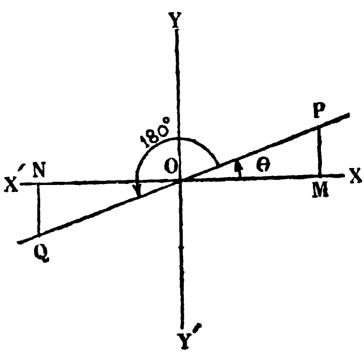

Fig. 24 (i)

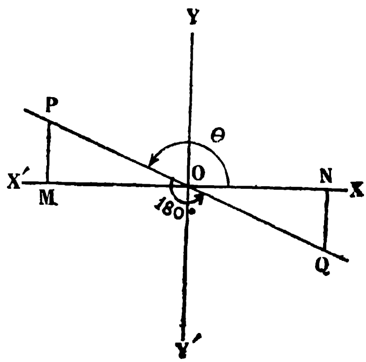

Fig. 24 (ii)

it further revolve in the same direction an angle $POQ = 180°$; so the $\angle XOQ = \angle 180° + \theta$ and OP and OQ are collinear.

Take OP=OQ and draw PM and QN perps. to OX (or OX′). Since $\angle POM = \angle QON$ in magnitude (vertically opposite

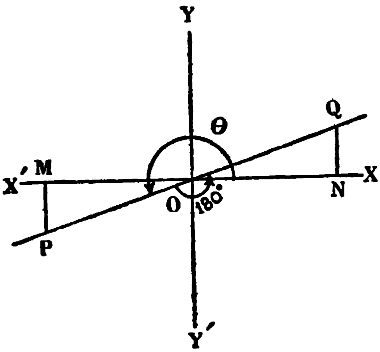

Fig. 24 (iii)

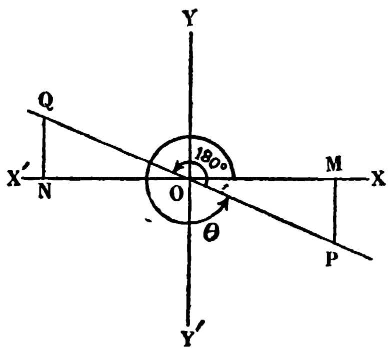

Fig. 24. (iv)

angles) and OP=OQ, the rt. angled $\triangle^s$ POM and QON are congruent. So, considering the signs as well,

$$QN=-PM, \quad ON=-OM, \quad OQ=OP.$$

$$\sin\ (180°+\theta)=\sin\ XOQ=\frac{QN}{OQ}=\frac{-PM}{OP}=-\sin\theta$$

$$\cos\ (180°+\theta)=\frac{ON}{OQ}=-\frac{OM}{OP}=-\cos\theta$$

$$\tan\ (180°+\theta)=\frac{QN}{ON}=\frac{-PM}{-ON}=\frac{PM}{OM}=\tan\theta$$

So, $\operatorname{cosec}\ (180°+\theta)=-\operatorname{cosec}\theta$, $\sec\ (180°+\theta)=\sec\theta$,
$\cot\ (180°+\theta)=\cot\theta$.

4-6. Ratios of $(270°\pm\theta)$

The students may proceed geometrically to find all the ratios of $270°\pm\theta$. Otherwise, their values may be deduced as follows :

$$\sin\ (270\pm\theta)=\sin\ (180+90°\pm\theta)=-\sin\ (90°\pm\theta)=-\cos\theta.$$

$$\cos\ (270\pm\theta)=\cos\ (180°+90°\pm\theta)=-\cos\ (90°\pm\theta)$$
$$=-(\mp\sin\theta)=\pm\sin\theta.$$

$$\tan\ (270°\pm\theta)=\frac{\sin\ (270°\pm\theta)}{\cos\ (270°\pm\theta)}=\frac{-\cos\theta}{\pm\sin\theta}=\mp\cot\theta.$$

So, $\operatorname{cosec}(270°\pm\theta)=-\sec\theta$

$\sec(270°\pm\theta)=\pm\operatorname{cosec}\theta$

$\cot(270°\pm\theta)=\pm\tan\theta$

Note: This procedure may also be adopted for angles $180°\pm\theta$.

4-7. Ratios of $(n.360°\pm\theta)$ where n is any integer

If n is an integer $n.\ 360°$ means n complete revolutions of the generating line. So the boundary line of $n.\ 360°+\theta$ is coincident with that of θ and of $n.\ 360°-\theta$ with that of $-\theta$. So the trigonometrical ratios of $n.\ 360°+\theta$ are the same as the corresponding ratios of θ, both in magnitude and in sign. And, similarly, the ratios of $n.\ 360°-\theta$ are the same as the corresponding ratios of $-\theta$, both in magnitude and in sign.

$$\sin(n.\ 360°-\theta)=\sin(-\theta)=-\sin\theta$$

$$\therefore\ \cos(n.\ 360°-\theta)=\cos(-\theta)=\cos\theta$$

$$\tan(n.\ 360°-\theta)=\tan(-\theta)=-\tan\theta$$

4-8. The following rule may be useful to remember the above results

I. If θ be associated with an even multiple of $90°$ (i. e. $180°\pm\theta$, $360°\pm\theta$) the ratio is not altered in form. That is, sine remains sine, cosine remains cosine and so on.

II. If, however, θ be associated with an odd multiple of $90°$ (i.e. $90°\pm\theta$, $270°\pm\theta$) the ratio is altered to the corresponding co-ratio. That is, the sine becomes cosine and vice versa, tangent becomes cotangent and vice versa and so on.

As regards the sign, determine the quadrant in which the associated angle lies (assume θ to be acute) and apply the rule "*all, sin, tan, cos*".

N. B. The angle $-\theta$ may be thought of as $0.360°-\theta$ and '0' may be taken as even in applying the above rule.

4-9. Certain trigonometrical ratios of a few important angles outside the first quadrant

$$\sin 120°=\sin 60°=\frac{\sqrt{3}}{2}$$

$$\cos 120°=-\cos 60°=-\frac{1}{2}$$

$$\sin 180°=\sin 0°=0$$

$$\cos 180°=-\cos 0°=-1$$

$$\tan 180°=0$$

4-10. Illustrated examples

Ex. 1. *Find sin $(540°+C)$*

$$\begin{aligned}\sin(540°+C) &= \sin(360°+180°+C)\\ &= \sin(180°+C)\\ &= -\cos C\end{aligned}$$

Ex. 2. *Find the value of cosec $(-210°)$ and cosec $(-1200°)$*

(a) $$\begin{aligned}\operatorname{cosec}(-210°) &= -\operatorname{cosec} 210°\\ &= -\operatorname{cosec}(180°+30°)\\ &= \operatorname{cosec} 30°\\ &= 2\end{aligned}$$

(b) $$\begin{aligned}\operatorname{cosec}(-1200°) &= -\operatorname{cosec} 1200°\\ &= -\operatorname{cosec}\{3\times 360°+120°\}\\ &= -\operatorname{cosec} 120°\\ \text{either} \quad &= -\operatorname{cosec}(90°+30°)\\ &= -\sec 30°\\ &= -\frac{2}{\sqrt{3}}\\ \text{or} \quad &= -\operatorname{cosec}(180°-60°)\\ &= -\operatorname{cosec} 60°\\ &= -\frac{2}{\sqrt{3}}\end{aligned}$$

N. B. We can always express the functions of any angle in terms of the functions of some positive acute angle. The procedure should be

(i) if the angle is negative, to use the relations connecting the functions of $-\theta$ and θ, e. g. Ex. 2 (a).

(ii) if the angle is numerically greater than 360°, to take off multiples of 360° and replace by a coterminal angle less than 360°, e.g. Ex. 2 (b).

(iii) if the angle is still greater than 180°, to use relations connecting functions of $180°+\theta$ and θ, e. g. Ex. 2 (a).

(iv) if the angle is still greater than 90°, to use the relations connecting functions of $90°+\theta$ and θ or, $(180°-\theta)$ and θ, e. g. Ex. 2 (b).

Ex. 3. *Prove that*

$$\sin(780^\circ)\cos(390^\circ)-\sin(330^\circ)\cos(-300^\circ)=1.$$

Left side $=\sin(2\times 360^\circ+60^\circ)\cos(360^\circ+30^\circ)$
$\quad -\sin(360^\circ-30^\circ)\cos(300^\circ)$

$=\sin 60^\circ \cos 30^\circ+\sin 30^\circ.\cos(360^\circ-60^\circ)$

$=\frac{\sqrt{3}}{2}.\ \frac{\sqrt{3}}{2}+\frac{1}{2}.\cos 60^\circ$

$=\frac{3}{4}+\frac{1}{2}\cdot\frac{1}{2}$

$=1$

Ex. 4. *Prove that*

$\sin\theta+\sin(\pi+\theta)+\sin(2\pi+\theta)+\ldots\ldots$ to n terms
$=0$ or $\sin\theta$ according as n is even or odd.

$\sin(\pi+\theta)=-\sin\theta$ $\qquad$ $\sin(2\pi+\theta)=\sin\theta$

$\sin(3\pi+\theta)=\sin(\pi+\theta)$ $\qquad$ $\sin(4\pi+\theta)=\sin\theta$

$=-\sin\theta$ $\qquad$ ....=etc.

.........=etc.

$\therefore$ Left side $=\sin\theta-\sin\theta+\sin\theta-\sin\theta+\ldots..$ to n terms
$=0$ or $\sin\theta$, according as n is even or odd.

Ex. 5. *Express in the simplest form*

$$\frac{\sin(-A)}{\sin(180^\circ+A)}+\frac{\cos A}{\sin(90^\circ-A)}-\frac{\tan(90^\circ+A)}{\cot A}$$

Expression $=\frac{-\sin A}{-\sin A}+\frac{\cos A}{\cos A}+\frac{\cot A}{\cot A}$

$=1+1+1$

$=3$

Ex. 6. *Solve for θ, giving all possible values, when $0^\circ<\theta<360^\circ$, the equation $\tan\theta=1$.*

Since $\tan\theta=1$

$\therefore$ $\theta=45^\circ$ is the smallest value satisfying the given equation. The tangent of an angle is also positive if the angle lies in the third quadrant.

$\therefore$ 2nd solution is $\theta=180^\circ+45^\circ=225^\circ$.

$\therefore$ $\theta=45^\circ,\ 225^\circ$

Examples 4

1. Find the values of

sin $(-870°)$, tan $(-855°)$, cos $(960°)$, cosec $(-660°)$, cot $(840°)$, sin $(930°)$.

2. Find the values of

$$\sin\frac{11\pi}{4},\ \tan\frac{16\pi}{3},\ \sec\left(\frac{3\pi}{2}+\frac{\pi}{4}\right).$$

3. When $\alpha=\frac{11\pi}{4}$, find the numerical value of

$$\sin^2\alpha-\cos^2\alpha+2\tan\alpha-\sec^2\alpha.$$

4. Solve for θ, numerically less than 360°, satisfying the following equations

(i) $\cos\theta=\frac{\sqrt{3}}{2}$ (v) $\operatorname{cosec}\theta=\frac{2}{\sqrt{3}}$

(ii) $\sin\theta=-\frac{1}{2}$ (vi) $\sec\theta=-\sqrt{2}$.

(iii) $\tan\theta=\frac{1}{\sqrt{3}}$

(iv) $\cot\theta=-1$

5. Simplify

(i) $\dfrac{\sin(180°-\theta)}{\tan(180°+\theta)}\cdot\dfrac{\cot(90°-\theta)}{\tan(90°+\theta)}\cdot\dfrac{\cos(360°-\theta)}{\sin(-\theta)}$

(ii) $\dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°+\theta)\sin(180°+\theta)\cot(90°-\theta)}$

(iii) $\dfrac{\operatorname{cosec}(180°-\theta)}{\sec(180°+\theta)}\cdot\dfrac{\cos(-\theta)}{\cos(90°+\theta)}$.

6. Prove that

(i) $\sin\{n\pi+(-1)^n\theta\}=\sin\theta$

(ii) $\tan\left(n\pi\pm\frac{\pi}{4}\right)=\pm1$, where n is any integer.

(iii) $\cos(n\pi+\alpha)=(-1)^n\cos\alpha$.

7. Evaluate the following

(i) $\sin^2 \frac{\pi}{4}+\sin^2 \frac{3\pi}{4}+\sin^2 \frac{5\pi}{4}+\sin^2 \frac{7\pi}{4}$

(ii) $\cos^2 \frac{\pi}{8}+\cos^2 \frac{3\pi}{8}+\cos^2 \frac{5\pi}{8}+\cos^2 \frac{7\pi}{8}$

(iii) $\cos \alpha+\cos (\pi+\alpha)+\cos (2\pi+\alpha)+\ldots\ldots$ to n terms.

(iv) $\tan \frac{\pi}{20} \cdot \tan \frac{3\pi}{20} \cdot \tan \frac{5\pi}{20} \cdot \tan \frac{7\pi}{20} \cdot \tan \frac{9\pi}{20}$

(v) $\dfrac{\cos \theta+\sin (-\theta)}{\sec \theta+\tan (-\theta)}$ when $\tan \theta=\frac{5}{12}$

8. Solve for θ, between 0° and 360°

(i) $6 \sin^2 \theta-5 \sin \theta-1=0$

(ii) $8 \sin^2 \theta+6 \cos \theta-9=0$

(iii) $4 \sin^2 \theta+3 \operatorname{cosec}^2 \theta=7$

(iv) $3 (\sec^2 \theta+\tan^2 \theta)=5$

(v) $2 (\cos x+\sec x)=5$

(vi) $\cot \theta-\tan \theta=\frac{2}{\sqrt{3}}$

(vii) $2 \sin \theta=\tan \theta$

9. If A, B, C are the angles of a triangle, show that

(i) $\sin (A+B)-\cos C=\cos (A+B)+\sin C$

(ii) $\tan (B+C)+\tan (C+A)+\tan (A+B)=\tan (\pi-A)$
$+\tan (2\pi-B)+\tan (3\pi-C)$

10. If A, B, C, D be the angles of a quadrilateral,

show that $\cos \frac{A+C}{2}+\cos \frac{B+D}{2}=0.$

If the above quadrilateral be inscribed in a circle,
show that $\Sigma \cos A=0.$

5

Addition And Subtraction Formulæ

5-1. To prove the formulæ

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

Let the $\angle XOY = A$, and $\angle YOZ = B$ where A and B are positive; then $\angle XOZ = A+B$.

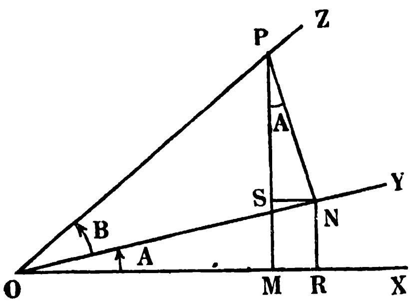

Fig. 25

On OZ, the boundary line of the angle $\overline{A+B}$ take any point P; draw perps. PM and PN to OX and OY respectively. Also draw perps. NR and NS to OX and PM in order.

By definition

$$\sin(A+B)=\frac{PM}{OP}=\frac{PS+SM}{OP}=\frac{PS+NR}{OP}=\frac{NR}{OP}+\frac{PS}{OP}$$
$$=\frac{NR}{ON}\cdot\frac{ON}{OP}+\frac{PS}{PN}\cdot\frac{PN}{OP}$$
$$=\sin A\cos B+\cos SPN\,.\,\sin B.$$

But $\angle SPN = 90° - \angle SNP = \angle SNO = \angle NOR = A$;

$$\therefore\quad \sin(A+B)=\sin A\cos B+\cos A\sin B \qquad \ldots\ldots(1)$$

Similarly, $\cos(A+B)=\frac{OM}{OP}=\frac{OR-MR}{OP}=\frac{OR-SN}{OP}=\frac{OR}{OP}-\frac{SN}{OP}$

$$=\frac{OR}{ON}\cdot\frac{ON}{OP}-\frac{SN}{PN}\cdot\frac{PN}{OP}$$
$$=\cos A.\cos B-\sin SPN.\sin B$$

or, $\cos(A+B)=\cos A\cos B-\sin A\sin B \qquad \ldots\ldots(2)$

5-2. To prove the formulæ

sin (A − B) = sin A cos B − cos A sin B
cos (A − B) = cos A cos B + sin A sin B

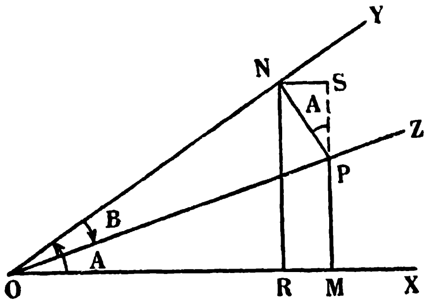

Fig. 26

Let the $\angle XOY = A$ and $\angle YOZ = B$ where A is positive, B negative and $A > B$; then $\angle XOZ = A - B$.

On OZ, the boundary line of the angle $\overline{A-B}$, take any point P; draw PM and PN perps. to OX and OY respectively. Also draw NR and NS perps. to OX and MP (produced) in order.

By definition

$$\sin (A-B) = \frac{PM}{OP} = \frac{SM-SP}{OP} = \frac{NR-SP}{OP} = \frac{NR}{OP} - \frac{SP}{OP}$$

$$= \frac{NR}{ON} \cdot \frac{ON}{OP} - \frac{SP}{NP} \cdot \frac{NP}{OP}$$

$$= \sin A \cos B - \cos SPN. \sin B$$

But $\angle SPN = 90° - \angle PNS = \angle SNY = \angle YOX = A$.

$\therefore$ $\sin (A-B) = \sin A \cos B - \cos A \sin B$ ......(3)

similarly, $\cos (A-B) = \frac{OM}{OP} = \frac{OR+RM}{OP} = \frac{OR+NS}{OP} = \frac{OR}{OP} + \frac{NS}{OP}$

$$= \frac{OR}{ON} \cdot \frac{ON}{OP} + \frac{NS}{NP} \cdot \frac{NP}{OP}$$

$$= \cos A \cos B + \sin SPN. \sin B$$

$\therefore$ $\cos (A-B) = \cos A \cos B + \sin A \sin B$ ......(4)

Note : The expansions of sin $(A \pm B)$ and cos $(A \pm B)$ are generally called the 'Addition and subtraction theorem'.

In the above proofs, the angles A, B, A+B are all less than 90° and A−B is positive. With certain modifications of the figures, the above theorems may be proved for angles of all magnitude.

We give below a proof of the theorems for angles of any magnitude, assuming the theorems to be true for acute angles.

Let A and B be two acute positive angles and $A+B<90^\circ$; now let $A'=90^\circ+A$ and $B'=B$.

$$\begin{aligned}\therefore\quad \sin(A'+B') &= \sin\{(90^\circ+A)+B\} = \sin(90^\circ+\overline{A+B})\\ &= \cos(A+B)\\ &= \cos A\cos B - \sin A\sin B \text{ (by Art. 1)}\\ &= \sin(90^\circ+A)\cos B + \cos(90^\circ+A)\sin B,\end{aligned}$$

since, $\sin(90^\circ+A)=\cos A$ and $\cos(90^\circ+A)=-\sin A$

$$\therefore\quad \sin(A'+B') = \sin A'\cos B' + \cos A'\sin B'$$

similarly for the other ratios.

Thus *the theorems are perfectly general.*

Caution ! $\sin(A\pm B)$ is *not equal to* $\sin A \pm \sin B$
$\cos(A\pm B)$ is *not equal to* $\cos A \pm \cos B$
and so on ;

This is because 'sin', 'cos' etc. are not numerical quantities ; rather only 'sin', only 'cos' bear no meaning unless they are associated with some angle.

$\sin(A+B)$ means 'sine' ratio of the angle $\overline{A+B}$. An example will illustrate the case. Let $A=30^\circ$, $B=60^\circ$

$$\begin{aligned}\therefore\quad \sin(A+B) &= \sin(30^\circ+60^\circ)\\ &= \sin 90^\circ = 1\end{aligned}$$

But, $\sin A+\sin B=\sin 30^\circ+\sin 60^\circ$

$$=\frac{1}{2}+\frac{\sqrt{3}}{2}=\frac{\sqrt{3}+1}{2}$$

5-3. Values of sin 75°, cos 75°, sin 15°, cos 15°

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ+30^\circ)\\ &= \sin 45^\circ\cos 30^\circ + \cos 45^\circ\sin 30^\circ\\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

$$\cos 75° = \cos(45°+30°)$$
$$= \cos 45° \cos 30° - \sin 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\therefore \quad \sin 15° = \cos(90° - 15°) = \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\cos 15° = \sin(90° - 15°) = \sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

N. B. sin 15° may as well be deduced by expanding sin (45°−30°) and similarly, cos 15° by expanding cos (45°−30°).

5-4. To prove

(I) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$
$= \cos^2 B - \cos^2 A$

(II) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$
$= \cos^2 B - \sin^2 A$

(I) Left side $= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$
$$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$$
$$= \sin^2 A - \sin^2 B$$
$$= (1-\cos^2 A) - (1-\cos^2 B)$$
$$= \cos^2 B - \cos^2 A$$

(II) Left side $= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$$
$$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$$
$$= \cos^2 A - \sin^2 B$$
$$= (1-\sin^2 A) - (1-\cos^2 B)$$
$$= \cos^2 B - \sin^2 A$$

5-5. Prove the formulæ

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \text{.........(5)}$$

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}},$$ dividing both numerator and denominator by cos A cos B;

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

similarly, it can be proved that

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

5-6. Prove the formulæ

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A}$$

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A} \qquad \ldots\ldots\ldots\ldots(6)$$

$$\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}$$

$$=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$$

$$=\frac{\frac{\cos A}{\sin A}\cdot\frac{\cos B}{\sin B}-1}{\frac{\cos B}{\sin B}+\frac{\cos A}{\sin A}},$$ dividing both numerator and denominator by sin A sin B,

$$=\frac{\cot A\cot B-1}{\cot B+\cot A}$$

Similarly, it can be proved that

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}$$

5-7. Values of tan 75° and tan 15°

$$\tan 75^\circ=\tan(45^\circ+30^\circ)$$

$$=\frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ.\tan 30^\circ}$$

$$=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$=\frac{(\sqrt{3}+1)\ (\sqrt{3}+1)}{3-1}$$ rationalising the denominator,

$$=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3}$$

$$\textbf{tan 15°}=\tan\ (45°-30°)$$

$$=\frac{\tan\ 45°-\tan\ 30°}{1+\tan 45°\ \tan\ 30°}$$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$=\frac{(\sqrt{3}-1)\ (\sqrt{3}-1)}{3-1}$$

$$=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}$$

Note: These values may as well be deduced by writing tan 75° = sin 75°/cos 75° etc. and utilising the values given in Art 3.

5-8. Prove the formulæ

$$\textbf{tan (45°+A)}=\frac{\mathbf{1+\tan A}}{\mathbf{1-\tan A}}$$

$$\textbf{tan (45°}-\textbf{A)}=\frac{\mathbf{1-\tan A}}{\mathbf{1+\tan A}} \qquad \text{.........(7)}$$

$$\tan\ (45°+A)=\frac{\tan\ 45°+\tan\ A}{1-\tan\ 45°\ \tan\ A}=\frac{1+\tan\ A}{1-\tan\ A}$$

$$\tan\ (45°-A)=\frac{\tan\ 45°-\tan\ A}{1+\tan\ 45°\ \tan\ A}=\frac{1-\tan\ A}{1+\tan\ A}$$

5-9. Expand (i) **sin (A+B+C)**

(ii) **cos (A+B+C)**

(iii) **tan (A+B+C)**

$$\begin{aligned}\sin(A+B+C) &= \sin(\overline{A+B}+C)\\ &= \sin(A+B)\cos C+\cos(A+B)\sin C\\ &= (\sin A\cos B+\cos A\sin B)\cos C\\ &\quad+(\cos A\cos B-\sin A\sin B)\sin C\\ &= \sin A\cos B\cos C+\cos A\sin B\cos C\\ &\quad+\cos A\cos B\sin C-\sin A\sin B\sin C\\ &= \cos A\cos B\cos C(\tan A+\tan B+\\ &\qquad \tan C-\tan A\tan B\tan C)\end{aligned}$$

$$\begin{aligned}\cos(A+B+C) &= \cos(\overline{A+B}+C)\\ &= \cos(A+B)\cos C-\sin(A+B)\sin C\\ &= (\cos A\cos B-\sin A\sin B)\cos C\\ &\quad-(\sin A\cos B+\cos A\sin B)\sin C\\ &= \cos A\cos B\cos C-\sin A\sin B\cos C\\ &\quad-\sin A\cos B\sin C-\cos A\sin B\sin C\\ &= \cos A\cos B\cos C(1-\tan A\tan B\\ &\qquad -\tan C\tan A-\tan B\tan C)\end{aligned}$$

$$\begin{aligned}\tan(A+B+C) &= \tan(\overline{A+B}+C)\\ &= \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}\\ &= \frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\tan B}\cdot\tan C}\end{aligned}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A.}$$

Otherwise :

$$\tan(A+B+C)=\frac{\sin(A+B+C)}{\cos(A+B+C)}$$

$$=\frac{\cos A\cos B\cos C(\tan A+\tan B+\tan C-\tan A\tan B\tan C)}{\cos A\cos B\cos C(1-\tan A\tan B-\tan B\tan C-\tan C\tan A)}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

5-10. Illustrated examples

Ex. 1. *If A, B are positive acute angles and $\cos A=\frac{3}{5}$ and $\sin B=\frac{5}{13}$, find $\cos(A-B)$ and $\sin(A+B)$*

$\because \quad \cos A=\frac{3}{5}, \ \sin A=\sqrt{1-(\frac{3}{5})^2}=\frac{4}{5}$

$\because \quad \sin B=\frac{5}{13}, \ \cos B=\sqrt{1-(\frac{5}{13})^2}=\frac{12}{13}$

$\therefore \quad \cos(A-B)=\cos A\cos B+\sin A\sin B$

$$=\frac{3}{5}\cdot\frac{12}{13}+\frac{4}{5}\cdot\frac{5}{13}$$

$$=\frac{36+20}{65}=\frac{56}{65}$$

and $\sin(A+B)=\sin A\cos B-\cos A\sin B$

$$=\frac{4}{5}\cdot\frac{12}{13}-\frac{3}{5}\cdot\frac{5}{13}$$

$$=\frac{48-15}{65}=\frac{33}{65}$$

Ex. 2. *Show that* $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}=\tan\alpha+\tan\beta$

$$\text{Left side}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$=\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}$$

$$=\tan\alpha+\tan\beta$$

Ex. 3. *Show that* $\tan(A+B)\tan(A-B)=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}$

$$\text{Left side}=\frac{\sin(A+B)}{\cos(A+B)}\cdot\frac{\sin(A-B)}{\cos(A-B)}$$

$$=\frac{(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)}{(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)}$$

$$=\frac{\sin^2 A\cos^2 B-\cos^2 A\sin^2 B}{\cos^2 A\cos^2 B-\sin^2 A\sin^2 B}$$

$$=\frac{\sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B}{\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B}$$

$$=\frac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}$$

Ex. 4. *If $A+B+C=\pi$ and $\cos A=\cos B \cos C$, show that $\tan A=\tan B+\tan C$*

$$\tan A=\frac{\sin A}{\cos A}=\frac{\sin (\pi-\overline{B+C})}{\cos B \cos C} \quad \text{(from given conditions)}$$

$$=\frac{\sin (B+C)}{\cos B \cos C}$$

$$=\frac{\sin B \cos C+\cos B \sin C}{\cos B+\cos C}$$

$$=\frac{\sin B}{\cos B}+\frac{\sin C}{\cos C}$$

$$=\tan B+\tan C$$

Ex. 5. *Show that* $\dfrac{\cos 8^\circ+\sin 8^\circ}{\cos 8^\circ-\sin 8^\circ}=\tan 53^\circ$

Right side $=\tan (45^\circ+8^\circ)$

$$=\frac{1+\tan 8^\circ}{1-\tan 8^\circ}$$

$$=\frac{1+\dfrac{\sin 8^\circ}{\cos 8^\circ}}{1-\dfrac{\sin 8^\circ}{\cos 8^\circ}}=\frac{\cos 8^\circ+\sin 8^\circ}{\cos 8^\circ-\sin 8^\circ}$$

Ex. 6. *If an angle θ is divided into two parts α, β such that $\tan \alpha : \tan \beta=x : y$, show that*

$$\sin (\alpha-\beta)=\frac{x-y}{x+y} \sin \theta.$$

We have, $\dfrac{x}{y}=\dfrac{\tan \alpha}{\tan \beta}=\dfrac{\sin \alpha \cos \beta}{\cos \alpha \sin \beta}$

$$\therefore \quad \frac{x-y}{x+y}=\frac{\sin \alpha \cos \beta-\cos \alpha \sin \beta}{\sin \alpha \cos \beta+\cos \alpha \sin \beta}$$

$$=\frac{\sin (\alpha-\beta)}{\sin (\alpha+\beta)}$$

$$=\frac{\sin (\alpha-\beta)}{\sin \theta} \quad (\because \alpha+\beta=\theta)$$

$$\therefore \quad \sin (\alpha-\beta)=\frac{x-y}{x+y} \cdot \sin \theta.$$

Examples 5

1. Given $\sin \alpha = \frac{1}{3}$, $\cos \beta = \frac{1}{8}$, find $\cos(\alpha+\beta)$; α, β are positive and acute

2. If $\sec A = \frac{17}{8}$ and $\operatorname{cosec} B = \frac{5}{4}$, find $\sec(A+B)$

3. Prove that $\dfrac{\sin(A-B)}{\sin A \sin B} = \cot B - \cot A$

4. Show that $\dfrac{\cos(\alpha-\beta)}{\sin \alpha \sin \beta} = \cot \alpha \cot \beta + 1$

5. Show that $\dfrac{\sin(\beta-\gamma)}{\cos\beta\cos\gamma} + \dfrac{\sin(\gamma-\alpha)}{\cos\gamma\cos\alpha} + \dfrac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} = 0$

6. Prove that

(i) $\sin(A+2B)\cos A + \cos(A+2B)\sin A - \sin(2A+2B) = 0$

(ii) $\sin A \cos(B+C) - \sin \cos(C+A) - \sin(A-B)\cos C = 0$

(iii) $\sin px \cos(p-1)x - \cos px \sin(p-1)x = \sin x$

7. Show that $\sin 2\theta . \cos\theta + \cos 2\theta \sin\theta = \sin 4\theta \cos\theta - \cos 4\theta \sin\theta$

8. Show that $\cot 2A + \tan A = \operatorname{cosec} 2A$

9. Show that $\tan^2 A - \tan^2 B = \dfrac{\sin(A+B)\sin(A-B)}{\cos^2 A \cos^2 B}$

10. Prove the following identities

(i) $\dfrac{\sin(A-B)}{\sin A \sin B} + \dfrac{\sin(B-C)}{\sin B \sin C} + \dfrac{\sin(C-A)}{\sin C \sin A} = 0$

(ii) $\dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C - \cos A} + \dfrac{\sin(A-B)}{\cos A \cos B} = 0$

(iii) $\dfrac{\tan A - \tan(A-B)}{1+\tan A \tan(A-B)} = \tan B$

(iv) $1 + \tan \alpha . \tan 2\alpha = \sec 2\alpha$

(v) $\tan(45^\circ + A) = \dfrac{\cos A + \sin A}{\cos A - \sin A}$

(vi) $\dfrac{1+\sin 2\theta}{1-\sin 2\theta} = \tan^2\left(\dfrac{\pi}{4} + \theta\right)$

(vii) $\tan(A+B) + \tan(A-B) = \dfrac{\sin 2A}{\cos^2 A - \sin^2 B}$

(viii) $\sec(x+y) = \dfrac{\sec x \sec y}{1 - \tan x \tan y}$

Prove that

11. $\dfrac{1-\tan^2(45^\circ-A)}{1+\tan^2(45^\circ-A)}=\sin 2A$

12. $\cos^6\alpha+\sin^6\alpha=1-\dfrac{3}{4}\sin^2 2\alpha$

13. $\dfrac{\cos A-\sin A}{\cos A+\sin A}=\sec 2A-\tan 2A$

14. $\dfrac{1}{\tan 3A-\tan A}-\dfrac{1}{\cot 3A-\cot A}=\cot 2A$

15. $\dfrac{1}{\tan 3\theta+\tan\theta}-\dfrac{1}{\cot 3\theta+\cot\theta}=\cot 4\theta$

16. $\tan 3A-\tan 2A-\tan A=\tan 3A.\ \tan 2A.\ \tan A$

17. $\dfrac{\cos 13^\circ+\sin 13^\circ}{\cos 13^\circ-\sin 13^\circ}=\tan 58^\circ$

18. If $\sin\alpha\sin\beta-\cos\alpha\cos\beta+1=0$, show that $1+\cot\alpha\tan\beta=0$

19. If $m\cos(x+\theta)=n\cos(x-\theta)$, show that $(m+n)\tan x=(m-n)\cot\theta$

20. If $A+B+C=\pi$, and $\cos A=\cos B\cos C$, show that $2\cot B\cot C=1$

21. If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$, show that $(\sin\alpha+\sin\beta+\sin\gamma)^2+(\cos\alpha+\cos\beta+\cos\gamma)^2=0$ and hence prove, $\Sigma\sin\alpha=0$ and $\Sigma\cos\alpha=0$

22. If $\sqrt{2}\cos\alpha=\cos\beta+\cos^3\beta$ and $\sqrt{2}\sin\alpha=\sin\beta-\sin^3\beta$, prove that $\sin(\alpha-\beta)=\pm\frac{1}{3}$

23. If $\dfrac{\tan(\alpha-\beta)}{\tan\alpha}+\dfrac{\sin^2\gamma}{\sin^2\alpha}=1$, prove that $\tan\alpha\tan\beta=\tan^2\gamma$

6

Transformation of Products and sums of Trigonometrical Ratios

6-1. Transformation of products into sums or differences.

From addition formulæ

$$\sin A \cos B + \cos A \sin B = \sin (A+B) \qquad \ldots\ldots(1)$$

$$\sin A \cos B - \cos A \sin B = \sin (A-B) \qquad \ldots\ldots(2)$$

Also,

$$\cos A \cos B - \sin A \sin B = \cos (A+B) \qquad \ldots\ldots(3)$$

$$\cos A \cos B + \sin A \sin B = \cos (A-B) \qquad \ldots\ldots(4)$$

Adding (1) and (2) : $\mathbf{2 \sin A \cos B = \sin(A+B) + \sin(A-B)}$

Subtract (2) from(1) : $\mathbf{2 \cos A \sin B = \sin(A+B) - \sin(A-B)}$

Adding (3) and (4): $\mathbf{2 \cos A \cos B = \cos(A+B) + \cos(A-B)}$

Subtract (3) from(4) : $\mathbf{2 \sin A \sin B = \cos(A-B) - \cos(A+B)}$

These formulæ enable us to transform

(i) the product of a sine and a cosine into the sum or difference of two sines.

(ii) the product of two cosines as the sum of two cosines.

(iii) the product of two sines as the difference of two cosines.

Note : The above formulæ may be conveniently remembered thus

$$2 \sin A \cos B = \sin\ (\text{sum}) + \sin\ (\text{diff.})$$
$$2 \cos A \sin B = \sin\ (\text{sum}) - \sin\ (\text{diff.})$$
$$2 \cos A \cos B = \cos\ (\text{sum}) + \cos\ (\text{diff.})$$
$$2 \sin A \sin B = \cos\ (\text{diff.}) - \cos\ (\text{sum})$$

Note that in the last formula, the 'diff.' precedes the 'sum'.

6-2. Transformation of sums or differences into products

Let $A+B=C$ and $A-B=D$.

$$\therefore \quad A = \frac{C+D}{2}, \quad B = \frac{C-D}{2}$$

Substituting for A, B in the set of formulæ of the previous article we at once obtain

$$\left.\begin{aligned}\sin C+\sin D&=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\\ \sin C-\sin D&=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\\ \cos C+\cos D&=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\\ \cos C-\cos D&=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\end{aligned}\right\}\ \ldots\ldots\ldots(2)$$

Note: This set of formulæ is of great importance in trigonometry and the students should master it thoroughly for further progress.

As an aid to memory we give the following verbal statement of the formulæ.

sum of two sines $=2$ sin ($\frac{1}{2}$ sum) cos ($\frac{1}{2}$ diff.)
diff. of two sines $=2$ cos ($\frac{1}{2}$ sum) sin ($\frac{1}{2}$ diff.)
sum of two cosines $=2$ cos ($\frac{1}{2}$ sum) cos ($\frac{1}{2}$ diff.)
diff. of two cosines $=2$ sin ($\frac{1}{2}$ sum) sin ($\frac{1}{2}$ diff. reversed)

6-3. Illustrated examples

Ex. 1. *Show that* $\sin 5\theta+\sin 2\theta-\sin\theta=\sin 2\theta(2\cos 3\theta+1)$

$$\begin{aligned}\text{Left side}&=(\sin 5\theta-\sin\theta)+\sin 2\theta\\ &=2\cos 3\theta.\sin 2\theta+\sin 2\theta\\ &=\sin 2\theta\,(2\cos 3\theta+1)\end{aligned}$$

Ex. 2. *Prove that* $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ=\frac{1}{16}$

(B. H. U. 1951)

$$\begin{aligned}\text{Left side}&=\tfrac{1}{2}\cos 20^\circ\cos 40^\circ\cos 80^\circ\quad(\because\ \cos 60^\circ=\tfrac{1}{2})\\ &=\tfrac{1}{4}\cos 20^\circ\,(2\cos 40^\circ.\cos 80^\circ)\\ &=\tfrac{1}{4}\cos 20^\circ\,(\cos 120^\circ+\cos 40^\circ)\\ &=\tfrac{1}{4}\cos 20^\circ\,(-\tfrac{1}{2}+\cos 40^\circ)\\ &=\tfrac{1}{8}\{2\cos 40^\circ\cos 20^\circ-\cos 20^\circ\}\\ &=\tfrac{1}{8}(\cos 60^\circ+\cos 20^\circ-\cos 20^\circ)\\ &=\tfrac{1}{8}\cos 60^\circ\\ &=\tfrac{1}{16}\quad(\because\ \cos 60^\circ=\tfrac{1}{2})\end{aligned}$$

Ex. 3. *Prove that* $\cos 20°+\cos 100°+\cos 140°=0$

$$\begin{aligned}\text{Left side}&=\cos 20°+(\cos 100°+\cos 140°)\\&=\cos 20°+2\cos 120°\cos 20°\\&=\cos 20°+2\left(-\tfrac{1}{2}\right)\cos 20°\\&=\cos 20°-\cos 20°\\&=0\end{aligned}$$

Ex. 4. *Show that* $\dfrac{\sin 7A-\sin 3A-\sin 5A+\sin A}{\cos 7A+\cos 3A-\cos 5A-\cos A}=\tan 2A$

$$\text{Left side}=\frac{(\sin 7A+\sin A)-(\sin 5A+\sin 3A)}{(\cos 7A-\cos A)-(\cos 5A-\cos 3A)}$$

$$=\frac{2\sin 4A\cos 3A-2\sin 4A\cos A}{2\sin 4A\sin(-3A)-2\sin 4A\sin(-A)}$$

$$=\frac{2\sin 4A(\cos 3A-\cos A)}{2\sin 4A(\sin A-\sin 3A)}$$

$$=\frac{2\sin 2A\sin(-A)}{2\cos 2A\sin(-A)}$$

$$=\tan 2A$$

Ex. 5. *Express* $4\cos\alpha\cos\beta\cos\gamma$ *as the sum of four cosines*

$$\begin{aligned}\text{Expression}&=2\cos\alpha.\{\cos(\beta+\gamma)+\cos(\beta-\gamma)\}\\&=2\cos\alpha.\cos(\beta+\gamma)+2\cos\alpha\cos(\beta-\gamma)\\&=\cos(\alpha+\beta+\gamma)+\cos(\alpha-\beta-\gamma)+\cos\\&\qquad(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma)\end{aligned}$$

Ex. 6. *If* $\sin x=k\sin y$, *prove that*

$$\tan\frac{x-y}{2}=\frac{k-1}{k+1}\tan\frac{(x+y)}{2}$$

We have, $\dfrac{\sin x}{\sin y}=k$

$$\therefore\quad\frac{\sin x-\sin y}{\sin x+\sin y}=\frac{k-1}{k+1}$$

$$\text{or,}\quad \frac{2\cos\frac{x+y}{2}.\ \sin\frac{x-y}{2}}{2\sin\frac{x+y}{2}\ \cos\frac{x-y}{2}}=\frac{k-1}{k+1}$$

$$\text{or,}\quad \cot\frac{x+y}{2}.\ \tan\frac{x-y}{2}=\frac{k-1}{k+1}$$

$$\text{or,}\quad \tan\frac{x-y}{2}=\frac{k-1}{k+1}\ \tan\frac{x+y}{2}$$

Ex. 7. *Prove that* $\sin^2 5x-\sin^2 3x=\sin 8x.\ \sin 2x$

Left side $=\sin(5x+3x).\ \sin(5x-3x)=\sin 8x\ \sin 2x$

[Vide Art. 5-4]

Ex. 8. *If* $\operatorname{cosec} A+\sec A=\operatorname{cosec} B+\sec B$, *show that*

$$\tan A\ \tan B=\cot\frac{A+B}{2}$$ [*P. U.* 1936]

We have, $\sec A-\sec B=\operatorname{cosec} B-\operatorname{cosec} A$

$$\text{or,}\quad \frac{1}{\cos A}-\frac{1}{\cos B}=\frac{1}{\sin B}-\frac{1}{\sin A}$$

$$\text{or,}\quad \frac{\cos B-\cos A}{\cos A\cos B}=\frac{\sin A-\sin B}{\sin A\sin B}$$

$$\text{or.}\quad \frac{2\sin\frac{A+B}{2}.\ \sin\frac{A-B}{2}}{\cos A\ \cos B}=\frac{2\cos\frac{A+B}{2}\ \sin\frac{A-B}{2}}{\sin A\ \sin B}$$

$$\text{or,}\quad \frac{\sin A\sin B}{\cos A\cos B}=\frac{\cos\frac{A+B}{2}}{\sin\frac{A+B}{2}}$$

$$\text{or,}\quad \tan A\tan B=\cot\frac{A+B}{2}$$

Examples 6

Prove that

1. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\dfrac{A+B}{2}$ [A. U. 1955]

2. $\sin\alpha-\sin 2\alpha+\sin 3\alpha=4\sin\dfrac{\alpha}{2}\cos\alpha\cos\dfrac{3\alpha}{2}$

3. $\sin A+2\sin 3A+\sin 5A=4\sin 3A\cos^2 A$

4. $\dfrac{\sin 5A-\sin 3A}{\cos 5A+\cos 3A}=\tan A$

5. $\dfrac{\sin 2\alpha+\sin 5\alpha-\sin\alpha}{\cos 2\alpha+\cos 5\alpha+\cos\alpha}=\tan 2\alpha$

6. $\dfrac{\cos 7\theta+\cos 3\theta-\cos 5\theta-\cos\theta}{\sin 7\theta-\sin 3\theta-\sin 5\theta+\sin\theta}=\cot 2\theta$

7. $\cos 20^\circ\cos 40^\circ\cos 80^\circ=\frac{1}{8}$

8. $\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ=\frac{3}{16}$ [P. U. 1942]

9. $\cos 80^\circ+\cos 40^\circ-\cos 20^\circ=0$

10. $\sin 78^\circ-\sin 18^\circ+\cos 132^\circ=0$

11. $\sin^2 5\theta-\sin^2 2\theta=\sin 7A\sin 3A$

12. $\dfrac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ+\sin 9^\circ}=\tan 54^\circ$ (use $\sin\theta=\cos\overline{90^\circ-\theta}$)

13. $\dfrac{\sin 40^\circ+\cos 10^\circ}{\cos 40^\circ-\sin 10^\circ}=\cot 20^\circ$

14. $\dfrac{\cos 10^\circ-\sin 10^\circ}{\cos 10^\circ+\sin 10^\circ}=\tan 35^\circ$

15. $\sin(\alpha+\beta+\gamma)+\sin(\alpha-\beta-\gamma)+\sin(\alpha+\beta-\gamma)$
$+\sin(\alpha-\beta+\gamma)=4\sin\alpha\cos\beta\cos\gamma$

16. $\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)$
$-\sin(\alpha+\beta+\gamma)=4\sin\alpha\sin\beta\sin\gamma$

17. $\cos(\beta+\gamma-\alpha)-\cos(\gamma+\alpha-\beta)+\cos(\alpha+\beta-\gamma)$
$-\cos(\alpha+\beta+\gamma)=4\sin\alpha\cos\beta\sin\gamma$

18, $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$$=4\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}$$

19. $\cos A-\cos(60^\circ+A)-\cos(60^\circ-A)=0$

20. $\sin A+\sin(60^\circ-A)-\sin(60^\circ+A)=0$

21. $\tan\frac{A+B}{2}+\tan\frac{A-B}{2}=\frac{2\sin A}{\cos A+\cos B}$

22. If $\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}$, *show that* $\tan\frac{\alpha+\beta}{2}=\frac{a+b}{a-b}\tan\frac{\alpha-\beta}{2}$

23. If $x\cos A+y\sin A=k=x\cos B+y\sin B$, show that

$$\frac{x}{\cos\frac{A+B}{2}}=\frac{y}{\sin\frac{A+B}{2}}=\frac{k}{\cos\frac{A-B}{2}}$$

24. If $\sin\theta+\sin\phi=a$ and $\cos\theta+\cos\phi=b$, show that

$$\cos\frac{\theta-\phi}{2}=\frac{1}{2}\sqrt{a^2+b^2},$$

$$\cos\frac{\theta+\phi}{2}=\frac{b}{\sqrt{a^2+b^2}},\text{ and}$$

$$\tan\frac{\theta-\phi}{2}=\pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

25. If $\frac{\cos x}{\cos y}=\frac{a}{b}$, prove that

$$a\tan x+b\tan y=(a+b)\tan\frac{x+y}{2}$$

26. If $A+B=90^\circ$, show that the greatest value of

$$\cos A\cos B=\tfrac{1}{2}$$

27. Show that

$$\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n=2\cot^n\frac{A-B}{2}$$

or zero, according as n is even or odd. [P. U. 1933]

7

The multiple and sub-multiple angles

If A be any angle, angles $2A$, $3A$ etc. are said to be the multiples of A and angles $A/2$, $A/3$ etc. the sub-multiples of A.

7-1. Trigonometrical ratios of angle 2A

$$\sin 2A = \sin (A+A)$$
$$= \sin A \cos A + \cos A \sin A$$

or, $$\boxed{\sin 2A = 2 \sin A \cos A} \qquad \ldots\ldots\ldots(1)$$

$$\cos 2A = \cos (A+A)$$
$$= \cos A . \cos A - \sin A \sin A$$

or,
$$\cos 2A = \cos^2 A - \sin^2 A \qquad \ldots\ldots\ldots(2)$$
$$= 2\cos^2 A - 1 \qquad \ldots\ldots\ldots(3)$$
$$\text{or, } = 1 - 2\sin^2 A \qquad \ldots\ldots\ldots(4)$$

From (3) and (4), we get, by transposition,

$$1 + \cos 2A = 2\cos^2 A \qquad \ldots\ldots\ldots(5)$$
$$1 - \cos 2A = 2\sin^2 A \qquad \ldots\ldots\ldots(6)$$

$\therefore$ By division, $$\boxed{\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A} \qquad \ldots\ldots\ldots(7)$$

$$\tan 2A = \tan (A+A)$$
$$= \frac{\tan A + \tan A}{1 - \tan A . \tan A}$$

or, $$\boxed{\tan 2A = \frac{2\tan A}{1-\tan^2 A}} \qquad \ldots\ldots\ldots(8)$$

7-2. To show that

(i) $\sin 2A = \dfrac{2 \tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $(1 \pm \sin 2A) = (\cos A \pm \sin A)^2$

$$\sin 2A = 2 \sin A \cos A$$
$$= \frac{2 \sin A}{\cos A} \cdot \cos^2 A$$
$$= 2 \tan A \cos^2 A$$
$$= \frac{2 \tan A}{\sec^2 A}$$

or $$\boxed{\sin 2A = \frac{2 \tan A}{1+\tan^2 A}} \qquad \ldots\ldots\ldots(9)$$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= \cos^2 A\,(1-\tan^2 A)$$
$$= \frac{1-\tan^2 A}{\sec^2 A}$$

or $$\boxed{\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}} \qquad \ldots\ldots\ldots(10)$$

$$1 \pm \sin 2A = (\cos^2 A + \sin^2 A) \pm \sin 2A$$
$$= \cos^2 A + \sin^2 A \pm 2 \sin A \cos A$$
$$= (\cos A \pm \sin A)^2 \qquad \ldots\ldots\ldots(11)$$

7-3. Trigonometrical ratios of angle 3A

$$\sin 3A = \sin (2A + A)$$
$$= \sin 2A \cos A + \cos 2A \sin A$$
$$= 2 \sin A \cos^2 A + (1 - 2 \sin^2 A) \sin A$$
$$= 2 \sin A\,(1 - \sin^2 A) + (1 - 2 \sin^2 A) \sin A$$

or $$\boxed{\sin 3A = 3 \sin A - 4 \sin^3 A} \qquad \ldots\ldots\ldots(12)$$

$$\cos 3A = \cos (2A + A)$$
$$= \cos 2A \cos A - \sin 2A \sin A$$
$$= (2 \cos^2 A - 1) \cos A - 2 \sin^2 A \cos A$$
$$= (2 \cos^2 A - 1) \cos A - 2\,(1 - \cos^2 A) \cos A$$

or $\boxed{\cos 3A = 4\cos^3 A - 3\cos A}$ ....(13)

$$\tan 3A = \tan(2A + A)$$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A.\ \tan A}$$

$$= \frac{\frac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \frac{2\tan A}{1-\tan^2 A}.\ \tan A}$$

or $\boxed{\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}}$ ....(14)

Note: The above formulæ are all perfectly general, that is, they are true for any value of A.

7-4. If in the above formulæ for multiple angles we put $A = \frac{\theta}{2}$ and $\frac{\theta}{3}$ respectively, we have the following set of formulæ for the submultiple angles

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= 2\cos^2\frac{\theta}{2} - 1$$

$$= 1 - 2\sin^2\frac{\theta}{2}$$

$$\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2}$$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$\sin\theta = \frac{2\tan\theta}{1+\tan^2\frac{\theta}{2}} \quad ......(15)$$

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$\sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3}$$

$$\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$$

$$\tan\theta = \frac{3\tan\frac{\theta}{3} - \tan^3}{1 - 3\tan^2\frac{\theta}{3}}$$

7-5. Given $\cos\theta$, to find $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$

Since, $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$$1 - 2\sin^2\frac{\theta}{2}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\tfrac{1}{2}(1-\cos\theta)} \quad \text{and ,}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\tfrac{1}{2}(1+\cos\theta)} \qquad \cdots\cdots(16)$$

Corresponding to one value of $\cos\theta$, therefore, there are two values of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$.

The presence of double value is explained thus : If $\cos\theta$ is given and nothing about θ be known, there is a series of values of θ and so of $\frac{\theta}{2}$. The angle $\frac{\theta}{2}$ may, therefore, lie in any quadrant and $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$ have the corresponding signs. The ambiguity of signs is removed if the quadrant to which θ belongs be known.

7-6. Given $\sin\theta$, to find $\sin\frac{\theta}{2}$, and $\cos\frac{\theta}{2}$

We have, $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$

and $2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta.$

By addition, $\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2 = 1 + \sin\theta$

and by subtraction, $\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2 = 1 - \sin\theta.$

$$\therefore \quad \sin\frac{\theta}{2}+\cos\frac{\theta}{2}=\pm\sqrt{1+\sin\theta}$$

$$\sin\frac{\theta}{2}-\cos\frac{\theta}{2}=\pm\sqrt{1-\sin\theta}$$

$$\therefore \quad \sin\frac{\theta}{2}=\pm\tfrac{1}{2}\sqrt{1+\sin\theta}\pm\tfrac{1}{2}\sqrt{1-\sin\theta}.$$

$$\text{and } \cos\frac{\theta}{2}=\pm\tfrac{1}{2}\sqrt{1+\sin\theta}\mp\tfrac{1}{2}\sqrt{1-\sin\theta}. \quad \text{.........}(17)$$

Since there is a double sign before each radical, there are **four** values for $\sin\frac{\theta}{2}$ and four values for $\cos\frac{\theta}{2}$ corresponding to **one** value of $\sin\theta$.

As before, the ambiguities of sign are due to the fact that when $\sin\theta$ is given and nothing about θ is known, there is a series of value of θ and consequently of $\frac{\theta}{2}$. The ambiguities may be removed if, in addition to the value of $\sin\theta$, we know that θ lies between certain limits.

7-7. Given $\tan\theta$, to find $\tan\frac{\theta}{2}$

$$\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$\text{or,} \quad \tan\theta\tan^2\frac{\theta}{2}+2\tan\frac{\theta}{2}-\tan\theta=0.$$

From this quadratic equation,

$$\tan\frac{\theta}{2}=\frac{-2\pm\sqrt{4+4\tan\theta.\tan\theta}}{2\tan\theta}$$

$$=\frac{-1\pm\sqrt{1+\tan^2\theta}}{\tan\theta}.$$

The presence of the double value may be explained as in the previous cases.

7.8. Trigonometrical ratios of $\frac{\theta}{3}$ from those of θ

We have, $\sin\theta=3\sin\frac{\theta}{3}-4\sin^3\frac{\theta}{3}$.

So, if $\sin\theta$ be given, we have a cubic equation to find $\sin\frac{\theta}{3}$. So corresponding to one value of $\sin\theta$, there are three values of $\sin\frac{\theta}{3}$,

Similarly, we may derive the values of $\cos\frac{\theta}{3}$ and $\tan\frac{\theta}{3}$ respectively from those of $\cos\theta$ and $\tan\theta$.

7-9. Ratios of 18°, 36°, 54°, 72°

Let $\theta=18°$; then $5\theta=90°$

$\therefore\quad 2\theta=90°-3\theta.$

$\therefore\quad \sin 2\theta=\cos 3\theta$

or, $2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta.$

or, $2\sin\theta=4\cos^2\theta-3$, Dividing by $\cos\theta$ ($\because\ \cos\theta\neq 0$).

$=4(1-\sin^2\theta)-3$

or, $4\sin^2\theta+2\sin\theta-1=0$

$$\therefore\quad \sin\theta=\frac{-2\pm\sqrt{4+16}}{8}=\frac{-1\pm\sqrt{5}}{4}$$

Since 18° is an acute angle, we take the +ve sign :

$$\therefore\quad \boxed{\sin 18°=\frac{\sqrt{5}-1}{4}} \qquad \text{.........(18)}$$

$$\therefore\quad \cos 18°=+\sqrt{1-\sin^2 18°}$$

$$=\sqrt{1-\frac{6-2\sqrt{5}}{16}}$$

$$\text{or.}\quad \cos 18°=\frac{1}{4}\sqrt{10+2\sqrt{5}} \qquad \text{.........(19)}$$

Since 72° and 18° are complementary,

$$\sin 72°=\cos 18°=\frac{1}{4}\sqrt{10+2\sqrt{5}}$$

$$\cos 72°=\sin 18°=\frac{\sqrt{5}-1}{4}$$

again, $\cos 36° = 1 - 2\sin^2 18°$

$$= \frac{1}{4}(\sqrt{5}+1)$$

$$\sin 36° = \sqrt{1-\cos^2 36°} = \frac{1}{4}(\sqrt{10-2\sqrt{5}})$$

Since 54° and 36° are complementary,

$$\sin 54° = \cos 36° = \frac{1}{4}(\sqrt{5}+1)$$

$$\cos 54° = \sin 36° = \frac{1}{4}(\sqrt{10-2\sqrt{5}})$$

7-10. Illustrated examples

Ex. 1. *Express cos 4A in terms of sin A*

$$\begin{aligned}\cos 4A &= 1-2\sin^2 2A\\ &= 1-2(4\sin^2 A\cos^2 A)\\ &= 1-8\sin^2 A(1-\sin^2 A)\\ &= 1-8\sin^2 A+8\sin^4 A\end{aligned}$$

Ex. 2. *If* $\tan x = b/a$, *find the value of* $a\cos 2x + b\sin 2x$

$$a\cos 2x + b\sin 2x = a.\frac{1-\tan^2 x}{1+\tan^2 x} + b.\frac{2\tan x}{1+\tan^2 x}$$

$$= a.\frac{1-\frac{a^2}{b^2}}{1+\frac{b^2}{a^2}} + b.\frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}}$$

$$= \frac{a\left(1-\frac{b^2}{a^2}\right)+2\frac{b^2}{a}}{1+\frac{b^2}{a^2}}$$

$$= \frac{a^2\left(1-\frac{b^2}{a^2}\right)+2b^2}{a\left(1+\frac{b^2}{a^2}\right)}$$

$$= \frac{a^2+b^2}{\frac{a^2+b^2}{a}}$$

$$= a$$

Ex. 3. *Prove that* $\sin^2 5x - \sin^2 3x = \sin 8x \sin 2x$

(of : Ex 7, Art 6-3)

$$\text{Left side} = (\sin 5x + \sin 3x)(\sin 5x - \sin 3x)$$
$$= (2 \sin 4x \cos x)(2 \cos 4x \sin x)$$
$$= (2 \sin 4x \cos 4x)(2 \sin x \cos x)$$
$$= \sin 8x.\ \sin 2x$$

Otherwise :

$$\sin 8x \sin 2x = \tfrac{1}{2}(\cos 6x - \cos 10x)$$
$$= \tfrac{1}{2}\{1 - 2\sin^2 3x - (1 - 2\sin^2 5x)\}$$
$$= \sin^2 5x - \sin^2 3x$$

Ex. 4. *Prove that* $\dfrac{\cos A - \sin A}{\cos A + \sin A} = \sec 2A - \tan 2A$

$$\text{Right side} = \frac{1}{\cos 2A} - \frac{\sin 2A}{\cos 2A}$$
$$= \frac{1 - \sin 2A}{\cos 2A}$$
$$= \frac{\cos^2 A + \sin^2 A - 2 \sin A \cos A}{\cos^2 A - \sin^2 A}$$
$$= \frac{(\cos A - \sin A)^2}{(\cos A + \sin A)(\cos A - \sin A)}$$
$$= \frac{\cos A - \sin A}{\cos A + \sin A}$$

Ex. 5. *If* $2 \tan \alpha = 3 \tan \beta$, *prove that*

$$\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$$ (C. U. 1946 ; A. U. 1949)

$\because\ 2 \tan \alpha = 3 \tan \beta \quad \therefore \quad \tan \alpha = \frac{3}{2} \tan \beta$

Now, $$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{\frac{3}{2}\tan \beta - \tan \beta}{1 + \frac{3}{2}\tan^2 \beta}$$

$$= \frac{\tan \beta}{2 + 3\tan^2 \beta} = \frac{\dfrac{\sin \beta}{\cos \beta}}{2 + 3\dfrac{\sin^2 \beta}{\cos^2 \beta}}$$

$$= \frac{2 \sin \beta \cos \beta}{4 \cos^2 \beta + 6 \sin^2 \beta} = \frac{\sin 2\beta}{4 \cos^2 \beta + 6(1 - \cos^2 \beta)}$$

$$=\frac{\sin 2\beta}{6-2\cos^2\beta}=\frac{\sin 2\beta}{5-(2\cos^2\beta-1)}$$

$$=\frac{\sin 2\beta}{5-\cos 2\beta}$$

Ex. 6. *If α and β are acute angles and* $\cos 2\alpha=\frac{3\cos 2\beta-1}{3-\cos 2\beta}$.

(C. U. 1941)

Show that $\tan\alpha=\sqrt{2}\tan\beta$

We have, $$\frac{1-\tan^2\alpha}{1+\tan^2\alpha}=\frac{\frac{3(1-\tan^2\beta)}{1+\tan^2\beta}-1}{3-\frac{1-\tan^2\beta}{1+\tan^2\beta}}$$

$$=\frac{2(1-2\tan^2\beta)}{2(1+2\tan^2\beta)}$$

$$=\frac{1-2\tan^2\beta}{1+2\tan^2\beta}$$

$\therefore$ By comp. and div.,

$$\frac{2\tan^2\alpha}{2}=\frac{4\tan^2\beta}{2}$$

or $\tan^2\alpha=2\tan^2\beta$

$\therefore$ $\tan\alpha=\pm\sqrt{2}\tan\beta$

$\because$ α, β are acute, only the positive value is to be taken.

Ex. 7. *Prove that* $\sec\theta+\tan\theta=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

[C. U, 1939, P. U. 1950]

Left side $$=\frac{1+\sin\theta}{\cos\theta}=\frac{\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$$

$$=\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

$$=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

Ex. 8. *If* $\tan\frac{\theta}{2}=\sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$, *show that*

$$\cos\phi=\frac{\cos\theta-e}{1-e\cos\theta}$$

[P. U. 1940, A. U. 1949]

We have, $\tan^2\frac{\theta}{2}=\frac{1-e}{1+e}\tan^2\frac{\phi}{2}$

$$\tan^2\frac{\phi}{2}=\frac{1+e}{1-e}\tan^2\frac{\theta}{2}$$

Now, $\cos\phi=\dfrac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}=\dfrac{1-\dfrac{(1+e)\tan^2\frac{\theta}{2}}{1-e}}{1+\dfrac{(1+e)\tan^2\frac{\theta}{2}}{1-e}}$

$$=\frac{1-e-(1+e)\tan^2\frac{\theta}{2}}{1-e+(1+e)\tan^2\frac{\theta}{2}}$$

$$=\frac{\left(1-\tan^2\frac{\theta}{2}\right)-e\left(1+\tan^2\frac{\theta}{2}\right)}{\left(1+\tan^2\frac{\theta}{2}\right)-e\left(1-\tan^2\frac{\theta}{2}\right)}$$

$$=\frac{\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}-e}{1-e\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}}$$

$$=\frac{\cos\theta-e}{1-e\cos\theta}$$

Ex. 9. *Show that*

$$\sin x = 2^n \cos\frac{x}{2} \cos\frac{x}{2^2} . \cos\frac{x}{2^3} \ldots\ldots\ldots\ldots \cos\frac{x}{2^n} \sin\frac{x}{2^n}$$

$$\sin x = 2 \sin\frac{x}{2} \cos\frac{x}{2}$$

$$\therefore \quad \sin\frac{x}{2} = 2 \sin\frac{x}{2^2} \cos\frac{x}{2^2}$$

$$\sin\frac{x}{2^2} = 2 \sin\frac{x}{2^3} \cos\frac{x}{2^3}$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$\sin\frac{x}{2^{n-1}} = 2 \sin\frac{x}{2^n} \cos\frac{x}{2^n}.$$

$$\therefore \quad \sin x = 2^n \cos\frac{x}{2} \cos\frac{x}{2^2} \cos\frac{x}{2^3} \ldots\ldots \cos\frac{x}{2^n} . \sin\frac{x}{2^n}$$

Examples 7

Prove the following identities

1. $\tan 2A - \sec A \sin A = \tan A \sec 2A$

2. $\dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta} = \tan\theta$ (C. U. 1938)

3. $\cos^6\alpha + \sin^6\alpha = 1 - \frac{3}{4}\sin^2 2\alpha$

4. $\dfrac{\sin\alpha - \sqrt{1+\sin 2\alpha}}{\cos\alpha - \sqrt{1+\sin 2\alpha}} = \cot\alpha$

5. $\dfrac{\sin A+\sin 2A}{1+\cos A+\cos 2A} = \tan A$

6. $4(\cos^3 20^\circ + \cos^3 40^\circ) = 3(\cos 20^\circ + \cos 40^\circ)$

7. $4(\cos^3 10^\circ + \sin^3 20^\circ) = 3(\cos 10^\circ + \sin 20^\circ)$

8. $\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$

9. $\dfrac{\sqrt{3}}{\sin 20^\circ}-\dfrac{1}{\cos 20^\circ}=4$

10. $\dfrac{1}{\sin 15^\circ}-\dfrac{1}{\cos 15^\circ}=2\sqrt{2}$

11. $\dfrac{1-\tan^2 (45^\circ-\theta)}{1+\tan^2 (45^\circ+\theta)}=\sin 2\theta$

12. $\cos^2(A-120^\circ)+\cos^2 A+\cos^2(A+120^\circ)=\frac{3}{2}$

13. $\dfrac{\sin 2^2\theta}{\sin\theta}=2^2 \cos\theta \cos 2\theta$

14. $\dfrac{\sin 2^n\theta}{\sin\theta}=2^n \cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta$

15. *If* $\theta=\dfrac{\pi}{2^n-1}$, *show that* $2^n \cos\theta.\ \cos 2\theta.\ \cos 2^2\theta\ldots$
$\ldots\ldots \cos^{n-1}\theta+1=0$ (A. U. 1953)

16. *Show that* $\dfrac{\tan 2^n\theta}{\tan\theta}=(1+\sec 2\theta)\ (1+\sec 2^2\theta)$
$(1+\sec 2^3\theta)\cdots\cdots\cdots\cdots\cdots(1+\sec 2^n\theta)$

17. *If* $\tan^2 x+2\tan x\tan 2y=\tan^2 y+2\tan y\tan 2x$, *prove that either* $\tan x=\pm\tan y$, *or* each side is unity

18. *If* $\tan^2 A=1+2\tan^2 B$, *show that* $\cos 2B=1+2\cos 2A$

19. *If* $\dfrac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)}=\dfrac{\tan\gamma}{\tan\beta}$, *show that* $\Sigma\sin 2\alpha=0$

20. *Show that* $\tan\left(\dfrac{\pi}{4}+\theta\right)-\tan\left(\dfrac{\pi}{4}-\theta\right)=2\tan 2\theta$

(A. U. 1957)

21. *Show that* $\tan\left(\dfrac{\pi}{4}+\theta\right)+\tan\left(\dfrac{\pi}{4}-\theta\right)=\dfrac{2}{\cos 2\theta}$

22. *If* $\tan\theta=\cos 2\beta$, *prove that* $\sin 2\theta=\dfrac{1-\tan^4\beta}{1+\tan^4\beta}$

(P. U. 1940)

23. Find sin $22\frac{1}{2}°$ and cos $22\frac{1}{2}°$ from sin 45°

24. Find sin 3° and cos 3°

Prove the following

25. $$\frac{1+\sin\theta}{\cos\theta}=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

26. $$\frac{\cos\theta}{1-\sin\theta}=\frac{\cot\frac{\theta}{2}+1}{\cot\frac{\theta}{2}-1}$$

27. $\sec A-\tan A=\tan\left(\frac{\pi}{4}-\frac{A}{2}\right)$

28. $\tan A+\sec A=\cot\left(\frac{\pi}{4}-\frac{A}{2}\right)$

29. $$\frac{1+\sin\theta}{1-\sin\theta}=\tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

30. $$\frac{\sin 2A}{1+\cos 2A}\cdot\frac{\cos A}{1+\cos A}=\tan\frac{A}{2}$$

31. $$\frac{\sin 2A}{1-\cos 2A}\cdot\frac{1-\cos A}{\cos A}=\tan\frac{A}{2}$$

32. $\tan 6°\tan 42°\tan 66°\tan 78°=1$ (B. H. U. 1954)

33. $\tan 7\frac{1}{2}°=\sqrt{6}-\sqrt{3}+\sqrt{2}-2$

34. $\cot 142\frac{1}{2}°=\sqrt{2}+\sqrt{3}-2-\sqrt{6}$

35. *If* $\tan\theta=\dfrac{\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}$, *show that one of the values of* $\tan\frac{\theta}{2}$ is $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$

36. *If* $\sin\alpha+\sin\beta=a$ and $\cos\alpha+\cos\beta=b$, *what is the value of* $\cos(\alpha+\beta)$?

37. Determine the limits between which A must lie in order that

(i) $2 \sin A = \sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}$

(ii) $2 \cos A = -\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}$

(iii) $2 \sin A = -\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}$

38. *Prove that* $2 \sin \frac{1}{2}A = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}$, and determine which are the correct signs when $270° > A > 180°$

(B. H. U. 1931)

39. If $A = 240°$, is the following statement correct ?

$$2 \sin \frac{A}{2} = \sqrt{1+\sin A} - \sqrt{1-\sin A}$$

If not, how must it be modified ?

40. *If* $A = 170°$ *prove that* $\tan \frac{A}{2} = \frac{-1-\sqrt{1+\tan^2 A}}{\tan A}$

8

General values and trigonometrical equations

8-1. If $\sin\theta=\frac{1}{2}$, what is θ? The immediate answer is $\theta=\pi/6$. But this is not the only solution; $\theta=\pi-\pi/6$ also satisfies the relation $\sin\theta=1/2$. In fact, all angles coterminal with these two values of θ satisfy the above equation.

Thus, although the trigonometrical ratio of a given angle is unique, the converse is not true; in fact, corresponding to a given trigonometrical ratio, there is an infinite number of angles. We shall now derive general expressions to include all angles, having the same trigonometrical ratio.

8-2. To find a general expression for all the angles having a given sine (or cosecant)

Let α be the smallest positive angle having a given sine. Let the angle XOP be α and XOP′ be $\pi-\alpha$, and let OP, OP′ be

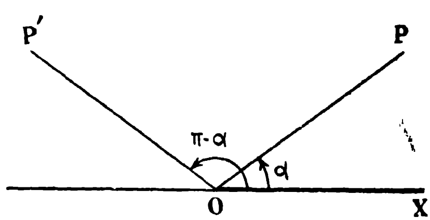

Fig. 27

their bounding lines. Then the required angles are those coterminal with OP and OP′.

The positive angles are, therefore, given by $2p\pi+\alpha$ and $2p\pi+(\pi-\alpha)$, where $p=0, 1, 2, 3,\cdots\cdots\cdots$so on.

The negative angles are given by $2q\pi-(\pi+\alpha)$ and $2q\pi-(2\pi-\alpha)$, where $q=0, -1, -2, -3\cdots\cdots$so on, since the angle XOP′ considered negatively is $-(\pi+\alpha)$and XOP, $-(2\pi-\alpha)$.

The above angles may be grouped as :

$$\left.\begin{matrix}2p\pi+\alpha\\(2q-2)\pi+\alpha\end{matrix}\right\}\text{ and }\left\{\begin{matrix}(2p+1)\pi-\alpha\\(2q-1)\pi-\alpha,\end{matrix}\right.$$

Noting that even multiples of π are followed by $+\alpha$ and odd multiples by $-\alpha$, the general expression for angles equisinal with α is

$$\boxed{\theta=n\pi+(-1)^n\alpha}$$

where $n=0, \pm1, \pm2, \pm3,\ldots\ldots\ldots$so on.

This is also the expression for all angles having the same cosecant as α.

Cor. If $\sin\theta=0$, then $\theta=n\pi$

8-3. To find a general expression for all the angles having a given cosine (or secant)

Let α be the smallest positive angle having a given cosine. Let the angle XOP be α and XOP′ be $2\pi-\alpha$ and let OP, OP′ be

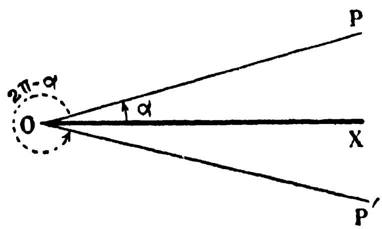

Fig. 28

their bounding lines. Then the required angles are those coterminal with OP and OP′.

The positive angles are, therefore, given by $2p\pi+\alpha$ and $2p\pi+(2\pi-\alpha)$, where $p=0, 1, 2, 3,\ldots\ldots$so on.

The negative angles are given by $2q\pi-\alpha$ and $2q\pi-(2\pi-\alpha)$, where $q=0,-1,-2,-3,\ldots\ldots$so on, since the angle XOP′, considered negatively, is $-\alpha$ and XOP $-(2\pi-\alpha)$.

The above angles may be grouped as :

$$\left.\begin{array}{l}2p\pi+\alpha\\2q\pi-\alpha\end{array}\right\}\text{ and }\left\{\begin{array}{l}(2p+2)\pi-\alpha\\(2q+2)\pi+\alpha\end{array}\right.$$

Noting that the multiples of π are always even but may by followed by $+\alpha$ or by $-\alpha$, the general expression for angles equi-cosinal with α is

$$\boxed{\theta=2n\pi\pm\alpha,}$$

where $n=0, \pm1, \pm2, \pm3,$......so on.

This is also the expression for all angles having the same secant as α.

Cor. **If $\cos\theta=1$, then $\theta=2n\pi$**

8-4. To find a general expression for all the angles having a given tangent (or cotangent)

Let α be the smallest positive angle having a given tangent. Let the angle XOP be α and XOP′ be $\pi+\alpha$ and let OP, OP′ be

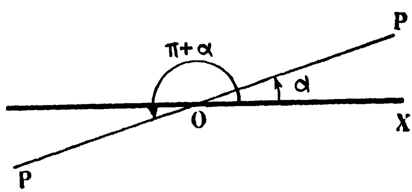

Fig. 29

their bounding lines. Then the required angles are those coterminal with OP and OP′.

The positive angles are, therefore, given by $2q\pi+\alpha$ and $2p\pi+(\pi+\alpha)$, where $p=0, 1, 2, 3,$......so on.

The negative angles are given by $2q\pi-(\pi-\alpha)$ and $2q\pi-(2\pi-\alpha)$, where $q=0,-1,-2,-3,$......so on. since the angle XOP′, considered negatively, is $-(\pi-\alpha)$ and XOP, $-(2\pi-\alpha)$.

The above angles may be grouped as :

$$\left.\begin{matrix}2p\pi+\alpha \\ (2q-2)\pi+\alpha\end{matrix}\right\} \text{and} \left\{\begin{matrix}(2p+1)\pi+\alpha \\ (2q-1)\pi+\alpha\end{matrix}\right.$$

Noting that the multiple of π, whether odd or even, is always followed by $+\alpha$, the general expression for angles equi-tangential with α is.

$$\boxed{\theta=n\pi+\alpha,}$$

where $n=0, \pm1, \pm2, \pm3, \ldots\ldots$so on.

This is also the expression for all angles having the same cotangent as α.

Cor. If $\tan\theta=0$, then $\theta=n\pi$

8-5. Illustrated examples

Ex. 1. *Find the general solution of* $\sin\theta=\frac{\sqrt{3}}{2}$

$\because \quad \sin\theta=\frac{\sqrt{3}}{2}$

$\therefore \quad \theta$ (least value) $=\frac{\pi}{3}$

$\therefore$ General solution is $\theta=n\pi+(-1)^n\frac{\pi}{3}$, n being an integer or zero.

Ex. 2. *Find the general solution of* $\tan^2\theta=3\ \operatorname{cosec}^2\theta-1$
(C. U. 1939)

By transposition, $\tan^2\theta+1=3\operatorname{cosec}^2\theta$

or, $\sec^2\theta=3\operatorname{cosec}^2\theta$

or, $\tan^2\theta=3$

$\therefore \quad \tan\theta=\pm\sqrt{3}$

From, $\tan\theta=\sqrt{3}=\tan\frac{\pi}{3}$, $\theta=n\pi+\frac{\pi}{3}$

From, $\tan\theta=-\sqrt{3}=\tan\left(-\frac{\pi}{3}\right)$, $\theta=n\pi-\frac{\pi}{3}$

$\therefore \quad \theta = n\pi \pm \frac{\pi}{3}$, n being any integer or zero

Ex. 3. *Solve* $\cos 9\theta = \cos 5\theta - \cos\theta$

By transposition, $(\cos 9\theta + \cos\theta) - \cos 5\theta = 0$

or, $2\cos 5\theta . \cos 4\theta - \cos 5\theta = 0$

or, $\cos 5\theta\,(2\cos 4\theta - 1) = 0$

$$\therefore \begin{cases} \text{Either, } \cos 5\theta = 0, \text{ whence } 5\theta = 2n\pi \pm \frac{\pi}{2} \text{ i.e. } \theta = \frac{(4n \pm 1)\pi}{10} \\ \text{or, } 2\cos 4\theta - 1 = 0, \text{ whence } 4\theta = 2n\pi \pm \frac{\pi}{3} \text{ i.e. } \theta = \frac{(6n \pm 1)\pi}{12} \end{cases}$$

Ex. 4. *Solve* $\tan x + \tan 2x + \tan x \tan 2x = 1$

(C. U. 1941, '45)

By transposition,

$$\tan x + \tan 2x = 1 - \tan x \tan 2x$$

or, $\dfrac{\tan x + \tan 2x}{1 - \tan x \tan 2x} = 1$

or, $\tan(x + 2x) = 1$

or, $\tan 3x = 1 = \tan\frac{\pi}{4}$

$\therefore \quad 3x = n\pi + \frac{\pi}{4}$

$\therefore \quad x = \frac{1}{3}\left(n\pi + \frac{\pi}{4}\right)$ where n is an integer or zero

Ex. 5. *Solve* $\cos m\theta = \sin n\theta$

Here, $\cos m\theta = \cos\left(\frac{\pi}{2} - n\theta\right)$

$\therefore \quad m\theta = 2k\pi \pm \left(\frac{\pi}{2} - n\theta\right)$, k is zero or any integer

$$\text{or, } \begin{cases} (m+n)\theta = (2k + \frac{1}{2})\,\pi \\ \text{and } (m-n)\theta = (2k - \frac{1}{2})\,\pi \end{cases}$$

The equation may as well be solved through the medium of sine, in which case, the result would be different in form.

Note : Corresponding to the different ways of obtaining the solution, the result may assume *different forms*.

Ex. 6. *Solve sin* $x+\sqrt{3}$ *cos* $x=\sqrt{2}$

(C. U. 1938, 47. B. H. U. 1950-'52)

Multiplying both sides by $\frac{1}{2}$, we get

$$\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x=\frac{1}{\sqrt{2}}$$

$$\therefore \quad \sin\frac{\pi}{6}\sin x+\cos\frac{\pi}{6}\cos x=\frac{1}{\sqrt{2}}$$

$$\therefore \quad \cos\left(x-\frac{\pi}{6}\right)=\frac{1}{\sqrt{2}}$$

$$\therefore \quad x-\frac{\pi}{6}=2n\pi\pm\frac{\pi}{4}$$

$$\therefore \quad x=2n\pi+\frac{5\pi}{12} \text{ or } 2n\pi-\frac{\pi}{12}$$

Note : In solving equations of this type i.e., $a\cos\theta+b\sin\theta=c$, it is objectionable to square the equation, for it introduces solutions not belonging to the given equation

$$\sin x+\sqrt{3}\cos x=\sqrt{2}$$

Re-arranging and squaring,

$$3\cos^2 x=(\sqrt{2}-\sin x)^2$$

But the solutions of this equation also contain those

$$-\sqrt{3}\cos x=\sqrt{2}-\sin x$$

In such cases, put $a=\gamma\cos\alpha$, $b=\gamma\sin\alpha$

$$\therefore \quad \gamma=\sqrt{a^2+b^2}, \text{ and}$$

$$\sin\alpha=\frac{b}{\sqrt{a^2+b^2}},$$

$$\cos\alpha=\frac{a}{\sqrt{a^2+b^2}},$$

$\therefore$ The equation $a \cos\theta + b \sin\theta = c$ becomes

$$\sqrt{a^2+b^2}\ \cos\alpha \cos\theta + \sqrt{a^2+b^2}\ \sin\alpha \sin\theta = c.$$

$$\text{or, } \cos(\theta-\alpha) = \frac{c}{\sqrt{a^2+b_2}} = \cos\beta, \text{ (say).}$$

$$\therefore\ \theta-\alpha = 2n\pi \pm \beta$$
$$\therefore\ \theta = 2n\pi + \alpha \pm \beta.$$

Ex. 7. *Solve cos* $2\theta = \cos\theta + \sin\theta$

From the eqn : $\cos^2\theta - \sin^2\theta = \cos\theta + \sin\theta$

or, $(\cos\theta+\sin\theta)(\cos\theta-\sin\theta) = \cos\theta + \sin\theta$
or, $(\cos\theta+\sin\theta)(\cos\theta+\sin\theta) = 0$

$\therefore$ either $\cos\theta + \sin\theta = 0$.........(1)
or. $\cos\theta - \sin\theta = 1$.........(2)

From (1) $\tan\theta = -1 \quad \therefore\ \theta = n\pi - \frac{\pi}{4}$

From (2), $\frac{1}{\sqrt{2}} \cos\theta - \frac{1}{\sqrt{2}} \sin\theta = \frac{1}{\sqrt{2}}$

$$\therefore\ \cos\theta \cos\frac{\pi}{4} - \sin\theta \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore\ \cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore\ \theta + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$$

$$\therefore\ \theta = 2n\pi \text{ or } 2n\pi - \frac{\pi}{2}$$

Ex. 8. *Solve tan* $p\theta = \tan q\theta$

we have $p\theta = n\pi + q\theta$

or, $(p-q)\,\theta = n\,\pi$

$$\therefore\ \theta = \frac{n\pi}{p-g}$$

Examples 8

Find the general solutions of

1. $2(\cos^2\theta-\sin^2\theta)=1$
2. $2\cos^2\theta+4\sin^2\theta=3$
3. $2\sin^2\theta-5\cos\theta-4=0$
4. $2\sin^2 x+\sin^2 2x=2$ (C. U. 1940)
5. $\sin 5\theta-\sin 3\theta=\sin\theta$
6. $\cos\theta-\cos 7\theta=\sin 4\theta$
7. $\tan x-\cot x=\operatorname{cosec} x$
8. $\tan\theta+\cot 2\theta=2$
9. $\sin p\theta+\sin q\theta=0$
10. $\sin p\theta+\cos q\theta=0$
11. $\sin 4\theta-\sin 3\theta+\sin 2\theta-\sin\theta=0$
12. $\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta=0$ (C. U. 1959)
13. $\tan^2\theta+\sec\theta=1$
14. $\cot^2\theta-1=\operatorname{cosec}\theta$
15. $\cot x-\tan x=2$ (C. U. 1934,37)
16. $\cos\theta-\sqrt{3}\sin\theta=1$
17. $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}$ (A. U. 1952)
18. $\cos\theta+\sqrt{3}\sin\theta=\sqrt{2}$ (C. U. 1944)
19. $\sin x+\cos x=\sqrt{2}$
20. $\sin\theta-\cos\theta=\sqrt{2}$ (A. U. 1949)
21. $3\cos\theta+\sqrt{3}\sin\theta=\sqrt{6}$ (B. H. U. 1949)
22. $\sin\theta+2\cos\theta=1$ (C. U. 1933)
23. $\sqrt{3}\cos x+\sin x=1$ (C. U. 1954)
24. $\cos x+\sin x=\cos 2x+\sin 2x$ (C. U. 1957, 1960)
25. $\tan x+\tan 2x+\tan 3x=0$ (A. U. 1941, 50. B. H. U. 1953)
26. $\tan x+\tan 2x+\tan 3x=\tan x\tan 2x\tan 3x$ (C. U. 1956)
27. $\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)=4$ (C. U. 1949)
28. $\cos 2x=\cos x\sin x$
29. $\cos^2\theta-\sin\theta-\frac{1}{4}=0$ (B. H. U. 1949)
30. $\cos 2x-\sin 2x=\cos x-\sin x-1$

31. $\sin\theta=\sqrt{2}\sin\phi$, $\sqrt{3}\cos\theta=\sqrt{2}\cos\phi$

32. $\operatorname{cosec}\theta=\sqrt{3}\operatorname{cosec}\phi$, $\cot\theta=3\cot\phi$

33. $\tan\theta+\tan 2\theta+\sqrt{3}\tan\theta\tan 2\theta=\sqrt{3}$ (B.H.U. 1951)

34. $\cos 3x+\cos 2x+\cos x=0$. (C.U. 1941, 46. P,U. 1941)

35. $\dfrac{\sin\alpha}{\sin 2x}+\dfrac{\cos\alpha}{\cos 2x}=2$ (C. U.)

36. $\cos x-\sin x=\cos\alpha+\sin\alpha$ (B. H. U. 1938)

37. $5\cos x+2\sin x=2$ (given, $\tan 21°42'=\cdot 4$)

38. If $\tan ax-\tan bx=0$, show that the values of x form a series in A. P.

39. Explain why the same two series of angles are given by the equations

$$\theta+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{6}$$

and

$$\theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{3}$$

40. If sin A = sin B, cos A = cos B, show that either A = B, or they differ by some multiple of four right angles.

(C. U. 1935)

9

Inverse circular functions

9-1. The statement '**sin θ=x**' means that '**θ is an angle whose sine ratio is x**'. Sometimes the above statement is given, for convenience, the symbol **θ=sin^{-1}x**. Thus, sin $\theta=x$ and $\theta=\sin^{-1}x$ are equivalent statements written only in two different forms ; if one of them be given, the other immediately follows. Similarly, $\cos^{-1}x$ means an angle whose cosine is x, $\tan^{-1}y$ signifies an angle whose tangent is y and so on.

Expressions $\sin^{-1}x$, $\cos^{-1}y$, $\tan^{-1}b$ etc. are called **inverse circular functions** to distinguish them from the sine, cosine etc., which are called the **circular functions.** '$\sin^{-1}x$' is read 'sine inverse x'.

Caution ! $\sin^{-1}x$ is not $(\sin x)^{-1}$ i.e. $1/\sin x$; for, $\sin^{-1}x$ denotes an angle and -1 here is not an algebraical index ; in $(\sin x)^{-1}$, however, $\sin x$ is a number and -1 is an algebraical index.

9-2. Principal values

$\text{Sin}^{-1}x$ has been defined as an angle whose sine ratio is x and not the angle whose sine is x ; in fact, if α be any angle whose sine is x, then all angles given by $n\pi+(-1)^n\alpha$, where $n=0, \pm1, \pm2, \ldots\ldots$so on, will have x for their sine.

An inverse function has, thus, an infinite number of values and of them, the smallest numerical (positive or negative) value is called its **principal value.** Thus, the principal value of $\sin^{-1}\frac{1}{2}$ is $\pi/6$.

In numerical examples, it is the principal value which is usually taken.

If, for a given ratio, there are two numerically equal angles —one positive and the other negative—the positive angle is taken as the principal value; eg. $\cos^{-1}\frac{1}{2}$ has the principal value $\frac{\pi}{3}$ and not $-\frac{\pi}{3}$.

9-3. The forms $\sin\sin^{-1}x$ and $\sin^{-1}\sin\theta$

It follows directly from the definition that $\sin^{-1}x$ is an angle whose sine is known to be x.

$$\therefore \quad \sin(\sin^{-1}x)=x$$

Similarly, $\sin^{-1}\sin\theta$ is an angle whose sine is $\sin\theta$; hence, $\sin^{-1}\sin\theta=\theta$

Otherwise : Let $\sin\theta=x$

$$\therefore \quad \theta=\sin^{-1}x$$

$$\therefore \quad \sin\sin^{-1}x=\sin(\theta)=x$$

$$\text{and } \sin^{-1}\sin\theta=\sin^{-1}(x)=\theta$$

Similarly, $\cos\cos^{-1}x=x,\ \tan\tan^{-1}x=x$

$\cos^{-1}\cos x=x,\ \tan^{-1}\tan x=x$ etc. etc.

9-4. *Show that* $\operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x}$

$$\sec^{-1}x=\cos^{-1}\frac{1}{x} \qquad \ldots\ldots\ldots (1)$$

$$\cot^{-1}x=\tan^{-1}\frac{1}{x}$$

Let $\operatorname{cosec}^{-1}x=\theta \quad \therefore \quad \operatorname{cosec}\theta=x$

$$\therefore \quad \sin\theta=\frac{1}{x} \text{ whence } \theta=\sin^{-1}\frac{1}{x}$$

$$\therefore \quad \operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x}$$

The other relations similarly follow.

9-5. *Show that* $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

$$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2} \qquad \ldots\ldots\ldots (2)$$

$$\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$$

Let $\sin^{-1}x=\theta \quad \therefore \quad \sin\theta=x$

But, $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$

$\therefore \quad \cos\left(\frac{\pi}{2}-\theta\right)=x$

$\therefore \quad \frac{\pi}{2}-\theta=\cos^{-1}x$

So, $\sin^{-1}x+\cos^{-1}x=\theta+\left(\frac{\pi}{2}-\theta\right)=\frac{\pi}{2}.$

The other relations similarly follow.

9-6. Show that

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy} \qquad \ldots\ldots\ldots(3)$$

$$\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}.$$

Let, $\tan^{-1}x=A \quad \therefore \quad \tan A=x$
and, $\tan^{-1}y=B \quad \therefore \quad \tan B=y$

Now, $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

$$=\frac{x+y}{1-xy}$$

$\therefore \quad A+B=\tan^{-1}\frac{x+y}{1-xy}$

or, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$

The other relation similarly follows.

Note : (i) If we put $y=x$, in the above relation,

$$2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2} \qquad \ldots\ldots\ldots(4)$$

(ii) It can be easily proved as above that

$$\cot^{-1}x \pm \cot^{-1}y = \cot^{-1}\frac{xy \mp 1}{y \pm x}$$

(iii) Remember : $\tan(\tan^{-1}x \pm \tan^{-1}y) = \dfrac{x \pm y}{1 \mp xy}$

9.7. **Show that**

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx} \quad \ldots(5)$$

Let $\tan^{-1}x = A \quad \therefore \quad \tan A = x$

$\tan^{-1}y = B \quad \therefore \quad \tan B = y$

$\tan^{-1}z = C \quad \therefore \quad \tan C = z$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$= \frac{x+y+z-xyz}{1-xy-yz-zx}$$

$$\therefore \quad A+B+C = \tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx}$$

or, $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\dfrac{x+y+z-xyz}{1-xy-yz-zx}$

Otherwise :

The left side $= \tan^{-1}\dfrac{x+y}{1-xy} + \tan^{-1}z$

$$= \tan^{-1}\frac{\dfrac{x+y}{1-xy} + z}{1 - \dfrac{x+y}{1-xy}.z}$$

= the right side.

9-8. Show that

$$\sin^{-1} x \pm \sin^{-1} y = \sin^{-1} \{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\} \quad \ldots\ldots(6$$

$$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1} \{xy \mp \sqrt{(1-x^2)(1-y^2)}\}$$

Let $\sin^{-1} x = A \quad \therefore \quad \sin A = x\,; \quad \cos A = \sqrt{1-x^2}$

$\sin^{-1} y = B \quad \therefore \quad \sin B = y\,; \quad \cos B = \sqrt{1-y^2}$

Now, $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$$= x\sqrt{1-y^2} \pm y\sqrt{1-x^2}$$

$\therefore \quad A \pm B = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$

or, $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1} \{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$

Similarly, the cosine relations follow from $\cos(A \pm B)$.

Note: By putting x for y, it follows from above that

$$2\sin^{-1} x = \sin^{-1} (2x\sqrt{1-x^2}) \quad \ldots\ldots\ldots(7)$$

$$2\cos^{-1} x = \cos^{-1} (2x^2 - 1)$$

9-9. Show that

$$2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2} \quad \ldots(8)$$

Let $\tan^{-1} x = \theta$, so that $\tan\theta = x$

From $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

we have $\sin 2\theta = \dfrac{2x}{1+x^2}$

or, $2\theta = \sin^{-1}\dfrac{2x}{1+x^2}$

or, $2\tan^{-1} x = \sin^{-1}\dfrac{2x}{1+x^2}$

The other relations follow from the following identities.

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

9·10. Illustrated examples

Ex. 1. *Show that* $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$$=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\frac{1}{\sqrt{1-x^2}}=\operatorname{cosec}^{-1}\frac{1}{x}$$

Let $\theta=\sin^{-1}x$ $\quad\therefore\quad \sin\theta=x$

$\therefore\quad \cos\theta=\sqrt{1-x^2},\ \tan\theta=\dfrac{x}{\sqrt{1-x^2}},\ \cot\theta=\dfrac{\sqrt{1-x^2}}{x}$

$\sec\theta=\dfrac{1}{\sqrt{1-x^2}},\ \operatorname{cosec}\theta=\dfrac{1}{x}$

$$\therefore\quad \theta=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$=\sec^{-1}\frac{1}{\sqrt{1-x^2}}=\operatorname{cosec}^{-1}\frac{1}{x}$$

Ex. 2. *Show that*

(i) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$

(ii) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$

(iii) $3\tan^{-1}x=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$. (C. U. 1938)

(i) Let $\theta=\sin^{-1}x$ $\quad\therefore\quad \sin\theta=x$

Now, $\sin 3\theta=3\sin\theta-4\sin^3\theta$
$=3x-4x^3$

$\therefore\quad 3\theta=\sin^{-1}(3x-4x^3)$

or, $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$

(ii) Let $\theta=\cos^{-1}x$ $\quad\therefore\quad \cos\theta=x$

Now, $\cos 3\theta=4\cos^3\theta-3\cos\theta$
$=4x^3-3x$

$\therefore\quad 3\theta=\cos^{-1}(4x^3-3x)$

or, $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$

(iii) Let $\theta=\tan^{-1} x \quad \therefore \quad \tan\theta=x$

$$\text{Now, } \tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

$$=\frac{3x-x^3}{1-3x^2}$$

$$\therefore \quad 3\theta=\tan^{-1}\frac{3x-x^3}{1-3x^2}$$

$$\text{or, } 3\tan^{-1} x=\tan^{-1}\frac{3x-x^3}{1-3x^2}.$$

Ex. 3. *Prove that* $\tan^{-1} 5-\tan^{-1} 3+\tan^{-1}\frac{7}{9}=\frac{\pi}{4}$.

$$\text{Left side}=\tan^{-1}\frac{5-3}{1+15}+\tan^{-1}\frac{7}{9}$$

$$=\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{7}{9}$$

$$=\tan^{-1}\frac{\frac{1}{8}+\frac{7}{9}}{1-\frac{7}{72}}$$

$$=\tan^{-1} 1=\frac{\pi}{4}$$

Ex. 4. *Prove that* (C. U. 1955)

$$\tan^{-1} a-\tan^{-1} c=\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}$$

$$\text{Right side}=(\tan^{-1}a-\tan^{-1}b)+(\tan^{-1}b-\tan^{-1}c)$$
$$=\tan^{-1} a-\tan^{-1}c$$

Ex. 5 *Show that* $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$ (C. U. 1937) (B.H.U. 1954)

$$\text{Left side}=\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}$$

$$=\tan^{-1} 1=\frac{\pi}{4}$$

Ex. 6. *Prove that* $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\dfrac{1-x}{1+x}$ (C. U. 1943)

Let $\tan^{-1}\sqrt{x}=\theta \quad \therefore \quad \tan\theta=\sqrt{x}$

Now, $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\dfrac{1-x}{1+x}$

$$\therefore \quad 2\theta=\cos^{-1}\frac{1-x}{1+x}$$

$$\therefore \quad \theta=\tfrac{1}{2}\cos^{-1}\frac{1-x}{1+x} \quad \text{Hence.}$$

Ex. 7. *If* $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$ *show that* $x+y+z=xyz$.

$$\text{Let } \left.\begin{aligned}\tan^{-1}x&=\alpha\\ \tan^{-1}y&=\beta\\ \tan^{-1}z&=\gamma\end{aligned}\right\} \qquad \therefore \quad \left.\begin{aligned}\tan\alpha&=x\\ \tan\beta&=y\\ \tan\gamma&=z\end{aligned}\right\}$$

$\therefore \quad \alpha+\beta+\gamma=\pi$ (from given condition)

$$\therefore \quad \alpha+\beta=\pi-\gamma$$

$$\therefore \quad \tan(\alpha+\beta)=-\tan\gamma$$

$$\text{or,} \quad \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=-\tan\gamma$$

or, $\dfrac{x+y}{1-xy}=-z$. Multiplying up and re-arranging,

$$x+y+z=xyz.$$

Ex. 8. *Solve the equation*

$$\tan^{-1}2x+\tan^{-1}3x=\frac{3\pi}{4}$$

We have, $\tan^{-1}\dfrac{2x+3x}{1-6x^2}=\dfrac{3\pi}{4}$

$$\therefore \quad \frac{2x+3x}{1-6x^2}=\tan\frac{3\pi}{4}=-1$$

$$\therefore \quad 6x^2-5x-1=0$$

$$\text{or,} \quad (6x+1)(x-1)=0$$

$$\therefore \quad x=1 \quad \text{or} -\tfrac{1}{6}$$

Ex. 9. *Solve the equation*

$$\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x$$ (C. U. 1947)

We have, $2\tan^{-1}a+2\tan^{-1}b=2\tan^{-1}x$

$$\left(\because \sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x\right)$$

$$\therefore \quad \tan^{-1}x=\tan^{-1}a+\tan^{-1}b$$

or, $$\tan^{-1}x=\tan^{-1}\frac{a+b}{1-ab}$$

or, $$x=\frac{a+b}{1-ab}$$

Examples 9

Prove that

1. $\sin^{-1}\frac{12}{13}=\cot^{-1}\frac{5}{12}$

2. $\tan^{-1}\frac{15}{8}=\operatorname{cosec}^{-1}\frac{17}{15}$

3. $\sec(\tan^{-1}x)=\sqrt{1+x^2}$

4. $\tan(\sec^{-1}x)=\sqrt{x^2-1}$

5. (i) $\tan^{-1}2+\tan^{-1}\frac{1}{2}=\frac{\pi}{2}$

   (ii) $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=1$ (B. H. U. 1949)

6. $\tan^{-1}\frac{2}{11}+\cot^{-1}\frac{24}{7}=\tan^{-1}\frac{1}{2}$

7. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{5}{6}+\tan^{-1}\frac{1}{11}$

8. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}=\cot^{-1}3$

9. (i) $2\tan^{-1}\frac{1}{3}+\sin^{-1}\frac{4}{5}=\frac{\pi}{2}$

   (ii) $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=\frac{\pi}{4}$ (B. H. U. 1950)

10. (i) $\tan^{-1} 1+\tan^{-1}2+\tan^{-1} 3=\pi=2(\tan^{-1} 1+\tan^{-1}\frac{1}{2}+\tan^{-1} \frac{1}{3})$

(ii) $\tan^{-1}\dfrac{1}{a+b}+\tan^{-1}\dfrac{b}{a^2+ab+1}=\tan^{-1}\dfrac{1}{a}$ (C. U. 1949)

11. $\tan^{-1} p-\tan^{-1} r=\tan^{-1}\dfrac{p-q}{1+pq}+\tan^{-1}\dfrac{q-r}{1+qr}$

12. $\cos^{-1} a+\cos^{-1} b=\sin^{-1}(a\sqrt{1-b^2}-b\sqrt{1-a^2})$

13. $\tan^{-1} x+\tan^{-1} y=\sin^{-1}\dfrac{x+y}{\sqrt{(1+x^2)(1+y^2)}}$

14. $\tan^{-1} \frac{1}{3}+\tan^{-1} \frac{1}{5}+\tan^{-1} \frac{1}{7}+\tan^{-1} \frac{1}{8}=\frac{\pi}{4}$ (C. U. 1942)

15. $\cos^{-1} x=\sin^{-1}\sqrt{\dfrac{1-x}{2}}+\cos^{-1}\sqrt{\dfrac{1+x}{2}}=2\sin^{-1}\sqrt{\dfrac{1-x}{2}}$

$=2\cos^{-1}\sqrt{\dfrac{1+x}{2}}$

16. $\sin^{-1}\sqrt{\dfrac{x-b}{a-b}}=\cos^{-1}\sqrt{\dfrac{a-x}{a-b}}=\tan^{-1}\sqrt{\dfrac{x-b}{a-x}}$

17. $\tan(2\tan^{-1} x)=2\tan(\tan^{-1} x+\tan^{-1} x^3)$

18. $\sec^2(\tan^{-1} 2)+\operatorname{cosec}^2(\cot^{-1} 3)=15$ (C. U. 1956)

19. $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$ (C. U. 1941)

20. $4(\cot^{-1} 3+\operatorname{cosec}^{-1}\sqrt{5})=\pi$ (C. U. 1939)

21. $\tan^{-1}\dfrac{b-c}{1+bc}+\tan^{-1}\dfrac{c-a}{1+ca}+\tan^{-1}\dfrac{a-b}{1+ab}=0$

22. If $\tan^{-1} a+\tan^{-1} b+\tan^{-1} c=\frac{\pi}{2}$ show that

$$bc+ca+ab=1$$

23. If $\cos^{-1} x+\cos^{-1} y+\cos^{-1} z=\pi$, prove that

$$x^2+y^2+z^2+2xyz=1$$

24. If $\sin^{-1} x \sin^{-1} y+\sin^{-1} z=\pi$, show that

$$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$$

25. If $\cos^{-1} x+\cos^{-1} y=\theta$, prove that

$$x^2-2xy\cos\theta+y^2=\sin^2\theta$$

26. If $u=\cot^{-1}\sqrt{\cos\alpha}-\tan^{-1}\sqrt{\cos\alpha}$, prove that

$$\sin u=\tan^2\frac{\alpha}{2}$$

27. Show that

(i) $\tan(\tan^{-1} x+\tan^{-1} y+\tan^{-1} z)$
$=\cot(\cot^{-1} x+\cot^{-1} y+\cot^{-1} z)$

(ii) $\tan^{-1}(\cot x)+\cot^{-1}(\tan x)=\pi-2x$

(iii) $\sin\operatorname{cosec}^{-1}\cot\tan^{-1} x=x$ (B. H. U. 1949)

(iv) $\cos\tan^{-1}\sin\cot^{-1} x=\sqrt{\dfrac{x^2+1}{x^2+2}}$ (A. U. 1950)

28. Solve the following equations :

(i) $\tan^{-1}(x+1)-\tan^{-1}(x-1)=\tan^{-1} 2$

(ii) $\cot^{-1} x+\cot^{-1} 2x=\dfrac{3\pi}{4}$

(iii) $\sin^{-1} x+\sin^{-1}(1-x)=\cos^{-1} x$

(iv) $\cos^{-1} x-\sin^{-1} x=\cos^{-1} x\sqrt{3}$

(v) $\cot^{-1}\dfrac{x-2}{x-1}+\cot^{-1}\dfrac{x+2}{x+1}=\dfrac{\pi}{4}$

(vi) $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\dfrac{\pi}{4}$

(vii) $\tan^{-1}\dfrac{2x}{1-x^2}+\cot^{-1}\dfrac{1-x^2}{2x}=\dfrac{\pi}{3}$

(viii) $\cos^{-1}\dfrac{1-a^2}{1+a^2}-\cos^{-1}\dfrac{1-b^2}{1+b^2}=2\tan^{-1} x$

(ix) $\cot^{-1}\dfrac{x^2-1}{2x}+\tan^{-1}\dfrac{2x}{x^2-1}+\dfrac{4\pi}{3}=0$

(x) $\tan^{-1} 2x+\tan^{-1} 3x=45°$ (A. U. 1949)

(xi) $\tan^{-1} 3x=\tan^{-1} x+\tan^{-1}(x+1)+\tan^{-1}(x-1)$.

(xii) $\tan^{-1}\dfrac{x+1}{x-1}+\tan^{-1}\dfrac{x-1}{x}=\tan^{-1}(-7)$ (B. H. U 1957)

29. If $\sin [2 \cos^{-1} \{\cot (2 \tan^{-1} x)\}]=0$, find x

30. If $\sin (\pi \cos \theta)=\cos (\pi \sin \theta)$, show that

$$2\theta=\pm \sin^{-1}\tfrac{3}{4}$$

31. If $\tan^{-1} x$, $\tan^{-1} y$, $\tan^{-1} z$ are in arthmetic progression, find the algebraic relation between x, y, z. If x, y, z are also in A. P. show that $x=y=z$

32. Find the values of

(i) $\sin \left(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right)$ (C. U. 1935)

(ii) $\tan \left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right)$

(iii) $\operatorname{cosec} (\tan^{-1} 2+\sec^{-1} 3)$

(iv) $\tan (\tan^{-1} x+\cot^{-1} x)$

33. Given $A+B+C=\pi$, show that

if $A=\tan^{-1} 2$, $B=\tan^{-1} 3$, then $C=\frac{\pi}{4}$ (C. U. 1951)

34. Show that

$\tan^{-1} (\frac{1}{2} \tan 2A)+\tan^{-1} (\cot A)+\tan^{-1} (\cot^3 A)=0$

(C. U. 1957)

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10

Trigonometrical identities

10-1. Many important identities involving the trigonometrical ratios of three or more angles can be established if there exists a relation among the angles. The most important of these trigonometrical identities are those in which the three angles are the angles of a triangle i.e. if A, B, C be the angles, $A+B+C=180°$. In proving these, we must keep clearly in view the properties of complementary and supplementary angles. Thus,

if $A+B+C=180°$,

we have,
$$\sin(B+C)=\sin A$$
$$\tan(C+A)=-\tan B$$
$$\sin C=\sin(A+B)$$
$$\cos(A+B)=-\cos C$$
$$\cos B=-\cos(C+A) \text{ and so on.}$$

Similarly, if $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90°$, we have,

$$\cos\frac{A+B}{2}=\sin\frac{C}{2}$$

$$\cos\frac{C}{2}=\sin\frac{A+B}{2}$$

$$\sin\frac{C+A}{2}=\cos\frac{B}{2}$$

$$\sin\frac{A}{2}=\cos\frac{B+C}{2}$$

$$\tan\frac{B+C}{2}=\cot\frac{A}{2} \text{ and so on.}$$

We establish below some such identities.

10-2 Illustrated examples

Ex. 1. *If $A+B+C=\pi$, prove that*

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$$

(C. U. 1953, A. U. 1950)

Left side $=2\sin(A+B)\cos(A-B)+2\sin C\cos C$

$=2\sin C.\cos(A-B)+2\sin C\cos C\ (\because A+B+C=\pi)$

$=2\sin C\{\cos(A-B)+\cos C\}$

$=2\sin C\{\cos(A-B)-\cos(A+B)\}\ (\because A+B+C=\pi)$

$=2\sin C.\ 2\sin A.\sin B$

$=4\sin A\sin B\sin C.$

Ex. 2. *If $A+B+C=\pi$, prove that*

$$\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1$$

Left side $=2\cos(A+B)\cos(A-B)+2\cos^2 C-1$

$=-2\cos C\cos(A-B)+2\cos^2 C-1\ (\because A+B+C=\pi)$

$=-2\cos C\{\cos(A-B)-\cos C\}-1$

$=-2\cos C\{\cos(A-B)+\cos(A+B)\}-1$

$(\because A+B+C=\pi)$

$=-2\cos C.\ \ 2\cos A.\cos B-1$

$=-4\cos A\cos B\cos C-1$

Ex. 3. *If $A+B+C=\pi$, Prove that* (C. U.)

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Left side $=2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}+2\sin\dfrac{C}{2}\cos\dfrac{C}{2}$

$=2\cos\dfrac{C}{2}\cos\dfrac{A-B}{2}+2\sin\dfrac{C}{2}\cos\dfrac{C}{2}$

$\left(\because \dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}=\dfrac{\pi}{2}\right)$

$=2\cos\dfrac{C}{2}\left\{\cos\dfrac{A-B}{2}+\sin\dfrac{C}{2}\right\}$

$=2\cos\dfrac{C}{2}\left\{\cos\dfrac{A-B}{2}+\cos\dfrac{A+B}{2}\right\}$

$\left(\because \dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}=\dfrac{\pi}{2}\right)$

$=2\cos\dfrac{C}{2}.\ 2\cos\dfrac{A}{2}\cos\dfrac{B}{2}$

$=4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

Ex. 4. *If $A+B+C=\pi$, prove that*

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(C. U. 1952)

Left side $=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}$

$$=2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}+1\left(\because \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\right)$$

$$=2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\sin\frac{C}{2}\right\}+1$$

$$=2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right\}+1$$

$$\left(\because \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\right)$$

$$=2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2}+1$$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Ex. 5. *If $A+B+C=\pi$, prove that*

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

(C. U. 1951, B. H. U. 1952, 55)

$\because \quad A+B=\pi-C$

$\therefore \quad \tan(A+B)=-\tan C$

or, $\dfrac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$

Multiplying up and re-arranging

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

Note: Otherwise, apply the formula of $\tan(A+B+C)$

Ex. 6. *If $A+B+C=\pi$, prove that*

$$\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$$

(C. U. 1936, 39. B. H. U. 1954, 56)

$\because \quad \frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$

$\tan\frac{A+B}{2}=\cot\frac{C}{2}$

$\therefore \quad \tan\left(\frac{A}{2}+\frac{B}{2}\right).\quad \tan\frac{C}{2}=1$

or, $\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}.\quad \tan\frac{C}{2}=1$

Multiplying up and re-arranging.

$$\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$$

Note: Otherwise, apply the formula $\tan\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)$

Ex. 7. *If $A+B+C=\pi$. prove that*

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$$

$$=1+4\sin\frac{B+C}{4}\sin\frac{C+A}{4}\sin\frac{A+C}{4}$$

(P. U. 1939, Sup. 41)

Right side$=1+2\sin\frac{\pi-A}{4}\left\{2\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}\right\}$

$=1+2\sin\frac{\pi-A}{4}\left\{\cos\frac{B-C}{4}-\cos\frac{2\pi-(B+C)}{4}\right\}$

$=1+2\sin\frac{\pi-A}{4}\left\{\cos\frac{B-C}{4}-\cos\frac{\pi+A}{4}\right\}$

$(\because \quad 2\pi-\overline{B+C}=\pi+\pi-\overline{B+C}=\pi+A)$

$$=1+2\sin\frac{\pi-A}{4}\cos\frac{B-C}{4}-2\sin\frac{\pi-A}{4}\cos\frac{\pi+A}{4}.$$

$$=1+2\sin\frac{B+C}{4}\cos\frac{B-C}{4}-\left(\sin\frac{\pi}{2}-\sin\frac{A}{2}\right)$$

$$=1+\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)-1+\sin\frac{A}{2}$$

$$=\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$$

Ex. 8. *If* $A+B+C=\pi$, *prove that*

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$$

$$=4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}$$

(B. H. U. 1951. P. U. 1944)

Right side$=2\cos\frac{\pi-A}{4}\left\{\cos\frac{2\pi-(B+C)}{4}+\cos\frac{B-C}{4}\right\}$

$$=2\cos\frac{\pi-A}{4}\cos\frac{\pi+A}{4}+2\cos\frac{\pi-A}{4}\cos\frac{B-C}{4}$$

$$=\left(\cos\frac{\pi}{2}+\cos\frac{A}{2}\right)+2\cos\frac{B+C}{4}\cos\frac{B-C}{4}$$

$$=\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}$$

Ex. 9. *If* $A+B+C=\pi$, *prove that* (P. U. 1945)

$$\sin^2A+\sin^2B+\sin^2C=2+2\cos A\cos B\cos C$$

Left side$=\frac{1}{2}(1-\cos 2A)+\frac{1}{2}(1-\cos 2B)+1-\cos^2C$

$=2-\frac{1}{2}(\cos 2A+\cos 2B)-\cos^2C$

$=2-\cos(A+B)\cos(A-B)-\cos^2C$

$=2+\cos C\cos(A-B)-\cos^2C$

$=2+\cos C\{\cos(A-B)-\cos C\}$

$=2+\cos C\{\cos(A-B)+\cos(A+B)\}$

$=2+\cos C.\ 2.\cos A\cos B$

$=2+2\cos A\cos B\cos C$

Ex. 10. *If* $A+B+C=\pi$, *prove that*

$$\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$$

(C. U. 1932, 37, 47, 59)

$$\begin{aligned}
\text{Left side} &= \tfrac{1}{2}(1+\cos 2A)+\tfrac{1}{2}(1+\cos 2B)+\cos^2 C \\
&= 1+\tfrac{1}{2}(\cos 2A+\cos 2B)+\cos^2 C \\
&= 1+\cos(A+B)\cos(A-B)+\cos^2 C \\
&= 1-\cos C\cos(A-B)+\cos^2 C \\
&= 1-\cos C\{\cos(A-B)-\cos C\} \\
&= 1-\cos C\{\cos(A-B)+\cos(A+B)\} \\
&= 1-\cos C.\ 2\cos A\cos B. \\
&= 1-2\cos A\cos B\cos C.
\end{aligned}$$

Ex. 11. *If* $A+B+C=\pi$, *prove that*

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

[A. U. 1954, 55, P. U. 1952 (Supp.)]

$$\begin{aligned}
\text{Left side} &= \frac{1}{2}(1-\cos A)+\frac{1}{2}+(1-\cos B)+\sin^2\frac{C}{2} \\
&= 1-\frac{1}{2}(\cos A+\cos B)+\sin^2\frac{C}{2} \\
&= 1-\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\sin^2\frac{C}{2} \\
&= 1-\sin\frac{C}{2}\cos\frac{A-B}{2}+\sin^2\frac{C}{2} \\
&= 1-\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\sin\frac{C}{2}\right\} \\
&= 1-\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right\} \\
&= 1-\sin\frac{C}{2}.\ 2\sin\frac{A}{2}\sin\frac{B}{2} \\
&= 1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}
\end{aligned}$$

Ex. 12. *If $A+B+C=\pi$, prove that*

$$\cot B \cot C+\cot C \cot A+\cot A \cot B=1$$

(C. U. 1955, B. H. U. 1957, A. U. 1953)

$\because \quad A+B=\pi-C$

$\therefore \quad \cot(A+B)=-\cot C$

or, $\dfrac{\cot A \cot B-1}{\cot B+\cot A}=-\cot C.$

Multiplying up and re-arranging,

$$\cot B \cot C+\cot C \cot A+\cot A \cot B=1$$

Ex. 13. *If $A+B+C=\pi$, prove that*

$$\frac{\cos A}{\sin B \sin C}+\frac{\cos B}{\sin C \sin A}+\frac{\cos C}{\sin A \sin B}=2$$

(C. U. 1949)

Left side $=\dfrac{\sin A \cos A+\sin B \cos B+\sin C \cos C}{\sin A \sin B \sin C}$

$=\dfrac{1}{2}\cdot\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A \sin B \sin C}$

$=\dfrac{1}{2}\cdot\dfrac{4 \sin A \sin B \sin C}{\sin A \sin B \sin C}$ [see Ex. 1.]

$=2$

Ex. 14. *If $\alpha+\beta+\gamma=\frac{\pi}{2}$, prove that*

$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=1-2\sin\alpha\sin\beta\sin\gamma$. (C.U. 1943)

Same as Ex. 11, if $A=2\alpha$, $B=2\beta$, $C=2\gamma$

Ex. 15. *If $\alpha+\beta=\gamma$, show that*

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma$$

(C. U. 1940, P. U. 1943)

Left side $=\frac{1}{2}(1+\cos 2\alpha)+\frac{1}{2}(1+\cos 2\beta)+\cos^2\gamma$

$=1+\frac{1}{2}(\cos 2\alpha+\cos 2\beta)+\cos^2\gamma$

$=1+\cos(\alpha+\beta)\cos(\alpha-\beta)+\cos^2\gamma$

$=1+\cos\gamma\cos(\alpha-\beta)+\cos^2\gamma \qquad (\because \alpha+\beta=\gamma)$

$=1+\cos\gamma\{\cos(\alpha-\beta)+\cos\gamma\}$

$=1+\cos\gamma\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}$

$=1+\cos\gamma.\ 2\cos\alpha\cos\beta$

$=1+2\cos\alpha\cos\beta\cos\gamma$

Ex. 16. *If $x+y+z=xyz$, prove that*

$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}$$

Let $x=\tan A$, $y=\tan B$, $z=\tan C$;

$\therefore$ $\tan A+\tan B+\tan C=\tan A \tan B \tan C$

$\therefore$ $\tan (A+B+C)=0$

$\therefore$ $A+B+C=n\pi$, where n is an integer

$\therefore$ $\tan (3A+3B+3C)=0$

$\therefore$ $\tan 3A+\tan 3B+\tan 3C=\tan 3A \tan 3B \tan 3C$

But, $\tan 3A=\dfrac{3 \tan A-\tan^3 A}{1-3 \tan^2 A}=\dfrac{3x-x^3}{1-3x^2}$, etc. Hence the result.

Examples 10

If $A+B+C=\pi$, prove that

1. $\sin 2A-\sin 2B+\sin 2C=4 \cos A \sin B \cos C$
2. $\sin 2A-\sin 2B-\sin 2C=-4 \sin A \cos B \cos C$
3. $\cos 2A+\cos 2B-\cos 2C=1-4 \sin A \sin B \cos C$
4. $\sin A+\sin B-\sin C=4 \sin\frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$
5. $\sin A-\sin B+\sin C=4 \sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$ (A.U. 1951)
6. $\cos A-\cos B+\cos C=4 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}-1$
7. $\dfrac{\sin B+\sin C-\sin A}{\sin A+\sin B+\sin C}=\tan \frac{B}{2} \tan \frac{C}{2}$
8. $\dfrac{1+\cos A-\cos B+\cos C}{1+\cos A+\cos B-\cos C}=\tan \frac{B}{2} \cot \frac{C}{2}$
9. (i) $\tan 2A+\tan 2B+\tan 2C=\tan 2A \tan 2B \tan 2C$

   (ii) $\tan 3A \tan 2A \tan A=\tan 3A-\tan 2A-\tan A$

   (B. H. U. 1957)

10. $(\cot B+\cot C)(\cot C+\cot A)(\cot A+\cot B)$
$=\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$

11. $\dfrac{\cot B+\cot C}{\tan B+\tan C}+\dfrac{\cot C+\cot A}{\tan C+\tan A}+\dfrac{\cot A+\cot B}{\tan A+\tan B}=1$

12. $\cot \frac{A}{2}+\cot \frac{B}{2}+\cot \frac{C}{2}=\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$ (C.U.1957)

13. $\cos^2 2A+\cos^2 2B+\cos^2 2C=1+2 \cos 2A \cos 2B \cos 2C$
(C. U. 1960)

14. $\dfrac{\tan A+\tan B+\tan C}{(\sin A+\sin B+\sin C)^2}=\dfrac{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}{2 \cos A \cos B \cos C}$

15. (i) $\sin^2 \frac{A}{2}-\sin^2 \frac{B}{2}+\sin^2 \frac{C}{2}=1-2 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$

(ii) $\sin^2 A+\sin^2 B-\sin^2 C=2 \sin A \sin B \cos C$
(B. H. U 1949, P. U 1940)

16. $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

17. $\sin (B+C-A)+\sin (C+A-B)+\sin (A+B-C)$
$=4 \sin A \sin B \sin C$ (A. U. 1953)

18. $\sin (B+2C)+\sin (C+2A)+\sin (A+2B)$

$$=4 \sin \frac{B-C}{4} \sin \frac{C-A}{2} \sin \frac{A-B}{2}$$

[B. H. U. 1950, P. U. 1940 (Sup.)]

19. $\dfrac{\tan B+\tan C}{\tan A} \cdot \dfrac{\tan C+\tan A}{\tan B} \cdot \dfrac{\tan A+\tan B}{\tan C}$
$=\sec A \sec B \sec C$

20. $\sin \dfrac{A+B}{2}+\sin \dfrac{B+C}{2}+\sin \dfrac{C+A}{2}$

$$=4 \cos \frac{A+B}{4} \cos \frac{B+C}{4} \cos \frac{C+A}{4}$$

21. $\cos\frac{A}{2}\cos\frac{B-C}{2}+\cos\frac{B}{2}\cos\frac{C-A}{2}+\cos\frac{C}{2}\cos\frac{A-B}{2}$
$=\sin A+\sin B+\sin C$

22. $\sin A\cos B\cos C+\sin B\cos C\cos A$
$+\sin C\cos A\cos B=\sin A\sin B\sin C$

23. $(\tan A+\tan B+\tan C)(\cot A+\cot B\cot C)$
$=1+\sec A\sec B+\sec C$

24. *If* $A+B+C=\pi$, and $\sin\left(A+\frac{C}{2}\right)=n\sin\frac{C}{2}$,

show that $\tan\frac{A}{2}\tan\frac{B}{2}=\frac{n-1}{n+1}$

If $A+B+C=\frac{\pi}{2}$, *prove that*

25. $\tan B\tan C+\tan C\tan A+\tan A\tan B=1$

26. $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin 2A+\sin 2B-\sin 2C}=\cot A\cot B$

27. $\frac{\tan B+\tan C}{\cot B+\cot C}+\frac{\tan C+\tan A}{\cot C+\cot A}+\frac{\tan A+\tan B}{\cot A+\cot B}=1$

28. (i) $\sin A+\sin B+\sin C=1+4\sin\frac{B+C}{2}\sin\frac{C+A}{2}$
$\sin\frac{A+B}{2}$

(ii) $\cot A+\cot B+\cot C=\cot A\cot B\cot C$
(P. U. 1943)

29. $\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C$
$\sin(A-B)=0$

30. *If* $\alpha=\beta+\gamma$, *show that* $\sin(\alpha+\beta+\gamma)+\sin(\alpha+\beta-\gamma)$
$+\sin(\alpha-\beta+\gamma)=4\sin\alpha\cos\beta\cos\gamma$

31. *If* A, B, C, D are the angles of a quadrilateral,

show that (i) $\frac{\tan A+\tan B+\tan C+\tan D}{\cot A+\cot B+\cot C+\cot D}$
$=\tan A\tan B\tan C\tan D.$

(ii) $\cos A+\cos B+\cos C+\cos D=4\cos\frac{A+B}{2}.$
$\cos\frac{B+C}{2}\cos\frac{C+A}{2}.$

32. *If* $\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-\beta)\sin(\gamma-\delta)$, *show that*

$$\cot\alpha\cot\beta\cot\gamma=\cot\delta$$

33. *If* $A+B+C=2S$ *show that*

(i) $\sin S(S-A)+\sin(S-B)\sin(S-C)=\sin B\sin C$

(ii) $\sin(S-A)+\sin(S-B)+\sin(S-C)-\sin S$

$$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(iii) $\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)$

$$=2+2\cos A\cos B\cos C$$ (A. U. 1956)

34. *If* $A+B+C=0$, *prove that*

$$\sin 2A+\sin 2B+\sin 2C=2(\sin A+\sin B+\sin C)$$
$$=(1+\cos A+\cos B+\cos C)$$

35. *If* $\sin A+\sin B+\sin C=0$, *prove that*

$$\sin 3A+\sin 3B+\sin 3C=-12\sin A\sin B\sin C$$

36. *If* $\cos A+\cos B+\cos C=0$, *prove that*

$$\cos 3A+\cos 3B+\cos 3C=12\cos A\cos B\cos C$$

(In Ex. 35 and 36, apply the formulæ of sin 3A and cos 3A in terms of A)

37. *If* $\cos(A+B)\sin(C+D)=\cos(A-B)\sin(C-D)$,

show that $\cot A\cot B\cot C=\cot D$

38. *Show that* $\tan(\beta-\gamma)+\tan(\gamma-\alpha)+\tan(\alpha-\beta)$

$$=\tan(\beta-\gamma)\tan(\gamma-\alpha)\tan(\alpha-\beta)$$

39. *Show that* $\cos^2(\beta-\gamma)+\cos^2(\gamma-\alpha)+\cos^2(\alpha-\beta)$

$$=1+2\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)$$

40. *If* $x+y+z=xyz$, *prove that*

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

41. *If* $A+B+C=\pi$, *prove that*

$$\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}=4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}$$

(P. U. 1942)

42. *If* $\alpha+\beta+\gamma=n\pi$, *show that*

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma=1$$

43. *If* $A+B+C=n\pi$ (n being zero or an integer),

show that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

11

Relations between the sides and angles of a triangle

In this chapter we shall denote the angles of a triangle by the letters A, B, C and the corresponding opposite side by a, b, c.

11-1. In any triangle, the sides are proportional to the sines of the opposite angles, that is,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Let ABC be any triangle and O be the centre and R be the radius of the circle circumscribing it.

Draw the diameter BD and join DC; so the $\angle$BCD is a rt. angle.

$\therefore$ From $\triangle$BCD, sin BDC $=\frac{BC}{BD}=\frac{a}{2R}$

But, the angle BDC=the angle A, as they rest on the same segment BC.

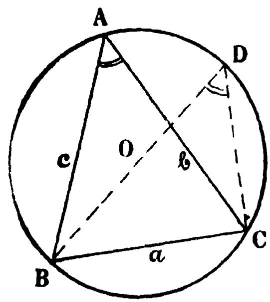

Fig. 30

$\therefore$ $\sin A=\frac{a}{2R}$. Similarly, it can be shown that

$$\sin B=\frac{b}{2R} \quad \text{and} \quad \sin C=\frac{c}{2R}$$

$$\therefore \quad \boxed{\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R} \qquad \text{.........(1)}$$

11-2. To express one side of a triangle in terms of the adjacent angles and the other two sides, that is,

$$\mathbf{a = b\cos C + c\cos B}$$
$$\mathbf{b = c\cos A + a\cos C}$$
$$\mathbf{c = a\cos B + b\cos A}$$

Let ABC be a triangle. In fig (i) the triangle is acute-angled

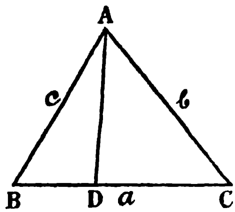

Fig. 31

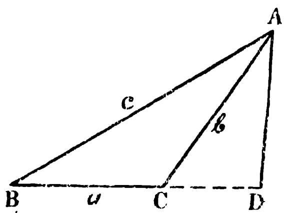

Fig. 32

at C and in (ii) obtuse-angled at C.

Draw AD perp. to BC (or produced). Then, from

figure (i) : $BC = BD + CD.$

$$= AB\cos ABD + AC\cos ACD$$

or, $a = c\cos B + b\cos C$

figure (ii) : $BC = BD - CD,$

$$= AB\cos ABD - AC\cos ACD$$
$$= c\cos B - b\cos(\pi - C)$$

or, $a = c\cos B + b\cos C$

Thus, in each case,
Similarly,

$$\boxed{\begin{aligned} a &= c\cos B + b\cos C \\ b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A \end{aligned}} \qquad \ldots\ldots(2)$$

11-3. To show, in any triangle,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \qquad \ldots\ldots(3)$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Take the figures of Art 2. Using a theorem of plane geometry, we have,

From fig (i) : $AB^2 = AC^2 + BC^2 - 2.\ BC.\ CD.$

or, $c^2 = b^2 + a^2 - 2a.\ b \cos C$

From fig (ii) : $AB^2 = AC^2 + BC^2 + 2\ BC.\ CD$

$= b^2 + a^2 + 2a.\ b \cos (\pi - C)$

or, $c^2 = b^2 + a^2 - 2\ ab \cos C$

Thus, in each case,

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Similarly, $$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

and $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

11-4. The three sets of formulæ deduced in the last three articles are not independent. Using the relation $A + B + C = \pi$ we may derive from any one set the other two.

We have, $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

since, $\sin A = \sin (B + C)$

$= \sin B \cos C + \cos B \sin C$

$$\therefore \quad 1 = \frac{\sin B}{\sin A} \cos C + \frac{\sin C}{\sin A} \cos B$$

$$= \frac{b}{a} \cos C + \frac{c}{a} \cos B$$

$\therefore \quad a = b \cos C + c \cos B$; Similarly,

$b = c \cos A + a \cos C$ and

$c = a \cos B + b \cos A$

Multiplying the above three equations respectively by a, b and c and adding.

$$a^2+b^2+c^2=2ab\cos C$$

$$\therefore\ \cos C=\frac{a^2+b^2-c^2}{2ab}\quad \text{Similarly,}$$

$$\cos B=\frac{c^2+a^2-b^2}{2ca}\quad \text{and}$$

$$\cos A=\frac{b^2+c^2-a^2}{2ab}$$

11-5. In any triangle prove that

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

In any triangle, $\frac{b}{c}=\frac{\sin B}{\sin C}$ By componendo and dividendo

$$\frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

$$=\frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$=\tan\frac{B-C}{2}\cot\frac{B+C}{2}$$

$$=\tan\frac{B-C}{2}\tan\frac{A}{2}\qquad\left[\because\ \frac{B+C}{2}=90^\circ-\frac{A}{2}\right]$$

$$\therefore\ \tan\frac{B-C}{2}=\frac{b-c}{b+c}\cdot\frac{1}{\tan\frac{A}{2}}$$

$$\text{or,}\qquad \tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\text{Similarly,}\qquad \tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}\qquad\ldots\ldots(4)$$

$$\text{and,}\qquad \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

11·6. Trigonometrical ratios of half angles in terms of the sides

We have $2\sin^2\frac{A}{2}=1-\cos A$

$$=1-\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a+b-c)(a-b+c)}{2bc}$$

Let now, $2s=a+b+c$, the perimeter of the triangle.

$$\therefore\quad a+b-c=2s-2c=2(s-c)$$

$$\text{and}\quad a-b+c=2s-2b=2(s-b)$$

$$\therefore\quad 2\sin^2\frac{A}{2}=\frac{4(s-c)(s-b)}{2bc}=\frac{2(s-b)(s-c)}{bc}$$

$\therefore\quad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$; since $\frac{A}{2}$ is always acute, only the positive value of the sq. root is taken.

Again, $2\cos^2\frac{A}{2}=1+\cos A$

$$=1+\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}$$

$$=\frac{2s.\ 2(s-a)}{2bc}$$

$$=\frac{2s(s-a)}{bc}$$

$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$; only the positive value of the sq. root is taken, since $\frac{A}{2}$ is always acute.

Also, $\tan\frac{A}{2}=\frac{\sin A/2}{\cos A/2}$

$$=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly, the ratios of $\frac{B}{2}$, $\frac{C}{2}$ can be derived.

Thus, we have,

$$\boxed{\begin{aligned} \sin\frac{A}{2}&=\sqrt{\frac{(s-b)(s-c)}{bc}}\\ \sin\frac{B}{2}&=\sqrt{\frac{(s-c)(s-a)}{ca}}\\ \sin\frac{C}{2}&=\sqrt{\frac{(s-a)(s-b)}{ab}}\end{aligned}} \qquad \boxed{\begin{aligned} \cos\frac{A}{2}&=\sqrt{\frac{s(s-a)}{bc}}\\ \cos\frac{B}{2}&=\sqrt{\frac{s(s-b)}{ca}}\\ \cos\frac{C}{2}&=\sqrt{\frac{s(s-c)}{ab}}\end{aligned}}$$

$$\boxed{\begin{aligned} \tan\frac{A}{2}&=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\\ \tan\frac{B}{2}&=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\\ \tan\frac{C}{2}&=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\end{aligned}} \qquad (5a, 5b, 5c)$$

11-7 To express sine of an angle in terms of the sides

$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$

$$=2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \quad \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Similarly,

$$\sin B = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)} \quad \ldots\ldots(6)$$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

Otherwise :

$$\sin^2 A = 1-\cos^2 A$$

$$=(1+\cos A)(1-\cos A)$$

$$=\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\left(1-\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\frac{(b+c)^2-a^2}{2bc}\cdot\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}\cdot\frac{(a+b-c)(a-b+c)}{2bc}$$

$$=\frac{2s.\ 2(s-a).\ 2(s-c).\ 2(s-b)}{4b^2c^2}$$

$$=\frac{4s\ (s-a)(s-b)(s-c)}{b^2c^2}.$$

$\therefore \quad \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$; only the positive value of the sq. root is taken, since sine of any angle of a triangle is always positive.

11-8. To find the area of a triangle

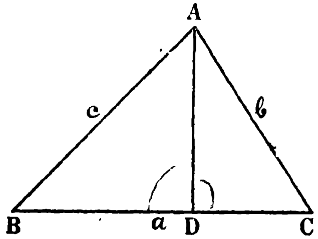

Fig. 33

Let $\triangle$ denote the area of the triangle ABC. Draw AD perp. to BC.

Now, $\triangle = \frac{1}{2}$. base $\times$ altitude

$= \frac{1}{2}$ BC. AD

$= \frac{1}{2}$ BC. AB sin B

$$\begin{array}{r|l|}
\therefore & \triangle = \frac{1}{2}\, ca \sin B \\
\text{Similarly} & \triangle = \frac{1}{2}\, bc \sin A \\
\text{and} & \triangle = \frac{1}{2}\, ab \sin C
\end{array} \quad \ldots\ldots(7)$$

Thus, area of a triangle $= \frac{1}{2}$ (product of two sides) $\times$ sine of the included angle.

Again, $\triangle = \frac{1}{2}\, bc \sin A$

$$= bc.\ \sin\frac{A}{2}\cos\frac{A}{2}$$

$$= bc.\sqrt{\frac{(s-b)(s-c)}{bc}}.\ \sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \quad \boxed{\triangle = \sqrt{s(s-a)(s-b)(s-c)}} \quad \ldots\ldots(8)$$

If we put $s = \frac{1}{2}(a+b+c)$,

$$\triangle = \tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}$$

or, $\triangle = \frac{1}{4}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}$ ......(9)

Again, $\triangle = \frac{1}{2}\, bc \sin A$

$$= \frac{1}{2}\, bc\, \frac{a}{2R} \qquad \left(\because \ \frac{a}{\sin A} = 2R\right)$$

or, $\boxed{\triangle = \frac{abc}{4R}}$ ......(10)

Note: If we use $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$, the formulæ of Art 7. become

$$\sin A = \frac{2\triangle}{bc}$$

$$\sin B = \frac{2\triangle}{ca} \qquad \ldots\ldots(11)$$

$$\sin C = \frac{2\triangle}{ab}$$

Those of (5c) become

$$\left.\begin{aligned}\tan\frac{A}{2}&=\frac{(s-b)(s-c)}{\Delta}\\ \tan\frac{B}{2}&=\frac{(s-c)(s-a)}{\Delta}\\ \tan\frac{C}{2}&=\frac{(s-a)(s-b)}{\Delta}\end{aligned}\right\}\quad\text{.........(11 a)}$$

Similarly, it can be derived that

$$\boxed{\begin{aligned}\cot\frac{A}{2}&=\frac{s(s-a)}{\Delta}\\ \cot\frac{B}{2}&=\frac{s(s-b)}{\Delta}\\ \cot\frac{C}{2}&=\frac{s(s-c)}{\Delta}\end{aligned}}\quad\text{.........(11 b)}$$

11-9. Circumradius of a triangle

In Art 1, we deduced,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

where R is the cricum-radius of the triangle.

$$\therefore\quad R=\frac{a}{2\sin A}=\frac{abc}{2\,bc\sin A}=\frac{abc}{4\Delta}$$

11-10. To find the in-radius of a triangle

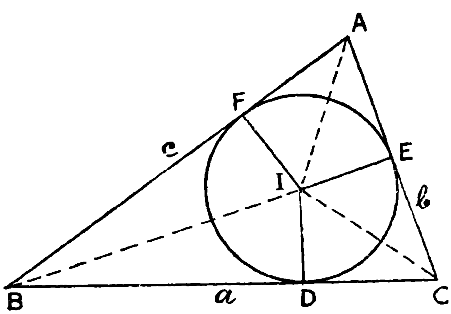

Fig. 34

Let I be the centre of the circle inscribed in the Δ ABC ;

D, E, F are the points of contact of the in-circle with the sides BC, CA, AB ; so ID, IE, IF are perps. to the sides and each of them is r, the in-radius.

$$\text{Now,}\quad \triangle=\triangle \mathrm{BIC}+\triangle \mathrm{CIA}+\triangle \mathrm{AIB}.$$
$$=\tfrac{1}{2}\,ar+\tfrac{1}{2}\,br+\tfrac{1}{2}\,cr$$
$$=\tfrac{1}{2}\,(a+b+c)\,r$$
$$=\mathbf{sr}\,;$$

$$\boxed{\therefore\quad \mathbf{r}=\triangle/\mathbf{s}}\qquad\ldots\ldots\ldots\,(12)$$

The in-radius may as well be expressed in terms of one side and the ratios of half-angles.

Since I is the in-centre, we have, from geometry,

$$\angle \mathrm{IBD}=\frac{\mathrm{B}}{2},\quad \angle \mathrm{ICD}\frac{\mathrm{C}}{2}.$$

$$\therefore\quad \mathrm{BD}=r\cot\frac{\mathrm{B}}{2},\quad \mathrm{CD}=r\cot\frac{\mathrm{C}}{2}.$$

$$\therefore\quad a=\mathrm{BC}=\mathrm{BD}+\mathrm{CD}$$
$$=r\left(\cot\frac{\mathrm{B}}{2}+\cot\frac{\mathrm{C}}{2}\right)$$
$$=r\,\frac{\cos\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}+\sin\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2}}{\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}}$$
$$=r\,\frac{\sin\frac{\mathrm{B}+\mathrm{C}}{2}}{\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}}$$

$$\therefore\quad r=\frac{a\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}}{\cos\frac{\mathrm{A}}{2}}\qquad\left(\because\quad \frac{\mathrm{B}+\mathrm{C}}{2}=90^\circ=\frac{\mathrm{A}}{2}\right)$$

Again since, $a=2\mathrm{R}\sin \mathrm{A}=4\mathrm{R}.\ \sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{A}}{2}$,

$$\boxed{\mathbf{r}=4\mathbf{R}\ \sin\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}.}\qquad\ldots\ldots\ldots(13)$$

Also, AE=AF, BD=BF and CD=CE

$\therefore$ $BD+CD+AF=\frac{1}{2}(a+b+c)=s$

$\therefore$ $AF=s-(BD+CD)=s-a$

But, $\dfrac{IF}{AF}=\tan\dfrac{A}{2}$

or, $r=(s-a)\tan\dfrac{A}{2}$

Similarly, $r=(s-b)\tan\dfrac{B}{2}$ ......(14)

$r=(s-c)\tan\dfrac{C}{2}$

Note : Distances of the in-centre from vertics are
$IA=r\operatorname{cosec}\frac{1}{2}A$, $IB=r\operatorname{cosec}\frac{1}{2}B$ and $IC=r\operatorname{cosec}\frac{1}{2}C$

11-11. To find the ex-radii of a triangle

Let I_1, be the centre of the escribed circle touching the side BC and the two sides AB and AC produced of the $\triangle ABC$

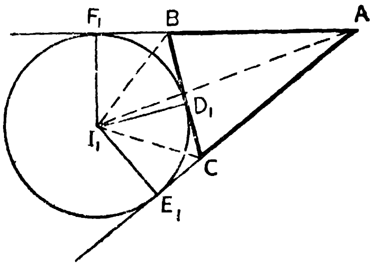

Fig. 35

respectively at D_1, F_1, and E_1. The lines I_1D_1, I_1E_1, I_1F_1. are, therefore, perps. to the sides.

Let r_1 be the radius of the ex-circle.

$\therefore$ $\triangle$ = area ABC
= area ABI_1C − area BI_1C
= (area BI_1A + area CI_1A) − area BI_1C
$=\frac{1}{2}cr_1+\frac{1}{2}br_1-\frac{1}{2}ar_1$
$=\frac{1}{2}(c+b-a)r_1$
$=(s-a)r_1$

$$\boxed{\therefore\quad r_1=\frac{\triangle}{s-a}} \qquad ......(15)$$

Similarly, if r_2, r_3 be the radii of the escribed circles opposite to the angles B and C respectively, we have

$$\text{and}\quad \boxed{\begin{aligned} r_2 &= \frac{\Delta}{s-b} \\ r_3 &= \frac{\Delta}{s-c} \end{aligned}} \qquad \ldots\ldots\ldots(16)$$

Since, again, I_1 is the point of intersection of the lines bisecting the angles B and C externally,

$$\therefore \quad \angle\, I_1BD_1 = 90^\circ - \frac{B}{2}$$

$$\angle\, I_1CD_1 = 90^\circ - \frac{C}{2}$$

$$\therefore \quad a = BD_1 + CD_1$$

$$= I_1D_1 \cot I_1BD_1 + I_1D_1 \cot I_1CD_1$$

$$= r_1 \tan \frac{B}{2} + r_1 \tan \frac{C}{2}$$

$$= r_1 \frac{\sin \frac{B+C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= r_1 \frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} \qquad \left(\because \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90^\circ\right)$$

$$\therefore \quad r_1 = a.\frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$$

$$\because \quad a = 2\,R \sin A = 2\,R \sin \frac{A}{2} \cos \frac{A}{2}$$

$$r_1 = 4\,R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

Similarly, $$r_2 = 4\,R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \qquad \ldots\ldots(17)$$

$$r_3 = 4\,R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

Also, $AE_1 = AC + CE_1 = b + CD_1$ ($\because$ $CE_1 = CD_1$)
and $AF_1 = AB + BF_1 = c + BD_1$ ($\because$ $BF_1 = BD_1$)

But, $AE_1 = AF_1$;

$\therefore$ By addition, $2AE_1 = b + c + BD_1 + CD_1$

$$= b + c + BC$$

$$= b + c + a = 2s$$

$$\therefore \quad AE_1 = s$$

Now, from $\triangle\ AI_1E_1$, $I_1E_1 = AE_1 \tan I_1AE_1$

$\therefore$ Similarly,

$$\boxed{\begin{aligned} r_1 &= s \tan \tfrac{1}{2} A \\ r_2 &= s \tan \tfrac{1}{2} B \\ r_3 &= s \tan \tfrac{1}{2} C \end{aligned}} \qquad \cdots \quad \cdots \quad (18)$$

Note : Distances of Ex-centres from the vertices are

$I_1A = r_1 \operatorname{cosec} \frac{1}{2} A = 4R \cos \frac{1}{2} B \cos \frac{1}{2} C$
$I_1B = r_1 \sec \frac{1}{2} B$; $I_1C = r_1 \sec \frac{1}{2} C$ Similarly,
$I_2B = r_2 \operatorname{cosec} \frac{1}{2} B$, $I_3C = r_3 \operatorname{cosec} \frac{1}{2} C$

11-12. Illustrated examples

Ex. 1. *Show that in any triangle,*

$$\sin A + \sin B + \sin C = \frac{s}{R} \qquad \text{(C. P. 1937)}$$

We have, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{1}{2R}$

$\therefore$ $\sin A = a/2R$; $\sin B = b/2R$; $\sin C = c/2R$.

$\therefore$ $\sin A + \sin B + \sin C = \dfrac{1}{2R}(a + b + c) = \dfrac{s}{R}$

where $a + b + c = 2s$

Ex. 2. *In any triangle, prove that*

$$a^2 + b^2 + c^2 = 2\,(bc \cos A + ca \cos B + ab \cos C)$$

We have, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$\left.\begin{aligned} \therefore\quad 2bc \cos A &= b^2 + c^2 - a^2 \\ \text{Similarly,}\quad 2ca \cos B &= c^2 + a^2 - b^2 \\ \text{and,}\quad 2ab \cos C &= a^2 + b^2 - c^2 \end{aligned}\right\}$$

$\therefore$ Adding, $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

Ex. 3. *In any triangle, prove that*

$$(b+c-a)\tan\frac{A}{2}=(c+a-b)\tan\frac{B}{2}=(a+b-c)\tan\frac{C}{2}$$

$$(b+c-a)\tan\frac{A}{2}=(b+c+a-2a)\tan\frac{A}{2}$$

$$=2(s-a)\tan\frac{A}{2}$$

$$=2(s-a).\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$=2.\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$=\frac{2}{s}\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\frac{2\triangle}{s}=2r$$

Similarly, we can show that

$$(c+a-b)\tan\frac{B}{2}=2r\ ;\quad (a+b-c)\tan\frac{C}{2}=2r$$

Otherwise :

$$(b+c-a)\tan\frac{A}{2}=2(s-a)\tan\frac{A}{2}$$

$$=2r \qquad \left[\because\ r=(s-a)\tan\frac{A}{2}\right]$$

Similarly, $(c+a-b)\tan\frac{A}{2}=2r$ etc. **Hence.**

Ex. 4. *If* $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}$, *show that the sides of the triangle are in A. P.* **(P. U. 1941)**

Since, $2a\cos^2\frac{C}{2}+2c\cos^2\frac{A}{2}=3b$

$\therefore\ a(1+\cos C)+c(1+\cos A)=3b$

or, $a+c+(a\cos C+c\cos A)=3b$

or, $a+c+b=3b$

$\therefore\ a+c=2b$

So, the sides *a, b, c* are in A. P.

Ex. 5. *Show that in any triangle,*

$$\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$$

(A. U. 1949. '56)

Left side $=\dfrac{b^2-c^2}{a^2}\cdot\ 2\sin A\cos A+\dfrac{c^2-a^2}{b^2}\cdot\ 2\sin B\cos B$

$$+\frac{a^2-b^2}{c^2}\,2\sin C\cos C$$

$$=\frac{1}{R}\left\{\frac{b^2-c^2}{a}\cdot\frac{b^2+c^2-a^2}{2bc}+\frac{c^2-a^2}{b}\cdot\frac{c^2+a^2-b^2}{2ca}\right.$$

$$\left.+\frac{a^2-b^2}{c}\cdot\frac{a^2+b^2-c^2}{2ab}\right\}$$

$$=\frac{1}{2\,abc\,R}\left\{(b^2-c^2)(b^2+c^2-a^2)+(c^2-a^2)(c^2+a^2-b^2)\right.$$

$$\left.+(a^2-b^2)\,(a^2+b^2-c^2)\right\}$$

$$=\frac{1}{2\,abc\ R}\times 0$$

$$=0$$

Ex. 6. *Show that*

$$\frac{bc-r_2r_3}{r_1}=\frac{ca-r_3r_1}{r_2}=\frac{ab-r_1r_2}{r_3}$$

We have $r_2r_3=\dfrac{\Delta}{s-b}\cdot\dfrac{\Delta}{s-c}$

$$=\frac{\Delta^2}{(s-b)(s-c)}$$

$=s(s-a)\qquad [\because\quad \Delta^2=s(s-a)(s-b)(s-c)]$

$\therefore\quad bc-r_2r_3=\frac{1}{4}\,[4bc-2s\,(2s-2a)]$

$=\frac{1}{4}\,[4bc-(a+b+c)\,(b+c-a)]$

$=\frac{1}{4}\,[4bc+a^2-(b+c)^2]$

$=\frac{1}{4}\,[a^2-(b-c)^2]$

$=\frac{1}{4}\,(a+b-c)(a-b+c)$

$=(s-b)\,(s-c)$

$$\therefore \quad \frac{bc - r_2 r_3}{r_1} = \frac{(s-b)(s-c)}{r_1} = \frac{(s-a)(s-b)(s-c)}{\Delta}$$

$$= \frac{\Delta}{s} \qquad \left(\because \quad r_1 = \frac{\Delta}{s-a}\right)$$

$$= r.$$

Similarly, it can be shown that the other ratios are also equal to r.

Ex. 7. *If* $\cos B = \dfrac{\sin A}{2 \sin C}$, *the triangle is isosceles.*

(A. U. 1953)

From the given condition, $\dfrac{c^2 + a^2 - b^2}{2ca} = \dfrac{a/2R}{2c/2R}$

$$\text{or,} \quad \frac{c^2 + a^2 - b^2}{a} = a$$

$$\text{or,} \quad c^2 + a^2 - b^2 = a^2$$

$$\text{or,} \quad c^2 - b^2 = 0$$

$$\therefore \quad c = b$$

Thus the triangle is isosceles.

Ex. 8. *If in any triangle,* $\cot \frac{A}{2}$, $\cot \frac{B}{2}$, $\cot \frac{C}{2}$ are in *A. P.* *show that a, b, c are also in A. P.*

We have, $\cot \frac{A}{2} - \cot \frac{B}{2} = \cot \frac{B}{2} - \cot \frac{C}{2}$

$$\text{or,} \quad \frac{s(s-a)}{\Delta} - \frac{s(s-b)}{\Delta} = \frac{s(s-b)}{\Delta} - \frac{s(s-c)}{\Delta}$$

$$\text{or,} \quad b - a = c - b$$

$\therefore$ a, b, c are in A. P.

Ex. 9. *Show that in any triangle,* $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} - \dfrac{1}{r} = 0$

(B. H. U. 1949, 50, 55. A. U. 1954)

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}$$

$$=\frac{3s-(a+b+c)}{\Delta}$$

$$=\frac{3s-2s}{\Delta}$$

$$=\frac{s}{\Delta}$$

$$=\frac{1}{r}$$

$$\therefore \quad \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r}=0$$

Ex. 10. *In any triangle, show that*

$$4\,(\cos A+\cos B+\cos C)=7, \quad \text{if } 3R=4r$$

From an well-known identity we have

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

where $A+B+C=\pi$.

Again $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$$\therefore \quad \frac{r}{R}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\therefore \quad \cos A+\cos B+\cos C=1+\frac{r}{R}$$

$$\therefore \quad 4\,(\cos A+\cos B+\cos C)=4\left(1+\frac{r}{R}\right)$$

$$=4\left(1+\frac{3}{4}\right) \quad [\because \quad 3R=4r]$$

$$=7$$

Examples 11

In any triangle ABC prove that

1. $(b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}$ (P. U. 1942)

2. $(b+c)\sin\frac{A}{2}=a\sin\left(\frac{A}{2}+B\right)$ (P. U. 1940)

3. $a(\sin B-\sin C)+b(\sin C-\sin A)$
$+c(\sin A-\sin B)=0$

4. $a(b\cos C-c\cos B)=b^2-c^2$

5. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=2s$

6. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$
[B. H. U. 1956. P. U. (Supp.) 1941]

7. $a\cos A+b\cos B+c\cos C=\frac{abc}{2R^2}$
$=4R\sin A\sin B\sin C$

8. $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0$
[P. U. (Supp.) 1944]

9. $a^2(\sin^2 B-\sin^2 C)+b^2(\sin^2 C-\sin^2 A)$
$+c^2(\sin^2 A-\sin^2 B)=0$

10. $a\cos B\cos C+b\cos C\cos A+c\cos A\cos B=\frac{\triangle}{R}$

11. $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$
(P. U. 1945)

12. $(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$

13. $\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$
(P. U. 1943)

14. $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)$
$=3abc$ (B. H. U. 1957. P. U. 1939)

15. $(b+c+a)\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=2a\cot\frac{A}{2}$
[P. U. (Supp) 1940]

16. $(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=2c\cot\frac{C}{2}$ (A. U. 1954)

17. $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=s^2$

18. $b^2\sin 2C+c^2\sin 2B=4\Delta$

19. $2R^2\sin A\sin B\sin C=\Delta$

20. $\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4\Delta}$

21. $\frac{a^2-b^2}{2}\cdot\frac{\sin A\sin B}{\sin(A-B)}=\Delta$

22. $\frac{\cos B\cos C}{bc}+\frac{\cos C\cos A}{ca}+\frac{\cos A\cos B}{ab}=\frac{1}{4R^2}$

23. $a^2\cot A+b^2\cot B+c^2\cot C=4\Delta$

24. $2\cot\frac{B}{2}+\cot\frac{A}{2}=\cot\frac{C}{2}$, if $2b=a+c$ (P. U. 1942)

25. *If* $C=120°$, *show that* $2c\cos\frac{A-B}{2}=\sqrt{3}(a+b)$

26. *If* $A=60°$, *show that* $2a\cos\frac{B-C}{2}=b+c$

27. *If* $C=60°$, *show that* $2a-b=2c\cos B$ (B. H. U. 1949)

28. *If* $A=90°$, *show that* $b+c=\sqrt{2}a\cos\frac{B-C}{2}$

29. *If* the cosines of two angles of a triangle are proportional to the opposite sides, *show that* the triangle is isosceles.

30. *If* the cosine of two angles of a triangle are inversely proportional to the opposite sides, *show that* the triangle is *either* isosceles *or* right angled.

31. In any triangle, *if* $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$

show that $C=60°$.

32. *If* $(a^2+b^2)\sin(A-B)=(a^2-b^2)\sin(A+B)$, *show that* the triangle is *either* isosceles *or* right angled.

33. *If* $b\tan B+c\tan C=(b+c)\tan\frac{B+C}{2}$, *show that* the triangle is isosceles.

34. *If* $\cos A+2\cos C : \cos A+2\cos B=\sin B : \sin C$, *show that* the triangle is *either* isosceles *or* right angled.

35. *If* cot A, cot B, cot C are in A.P., *show that* a^2, b^2, c^2 are also in A. P. (P. U. 1944)

36. *If* a, b, c are in A. P. *show that*

$$4(1-\cos A)(1-\cos C)=\cos A+\cos C$$

37. *If* $\sin A : \sin C=\sin(A-B) : \sin(B-C)$, *show that* a^2, b^2, c^2 are in A. P.

38. *If* $\cot A+\cot B+\cot C=\sqrt{3}$, *show that* the triangle is equiangular.

39. In a triangle *if* a^2, b^2, c^2 are in A. P., *show that* tan A, tan B, tan C are in H. P. (A. U. 1952)

40. *If* $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in A. P. *show that* cos A, cos B, cos C are also in A. P. (C. U. 1954)

41. Calculate the area of a triangle of sides 10, 12, 14.

42. A triangle has sides 3, 5, 7. Find the greatest angle.

Prove that in a triangle

43. $\cos A+\cos B+\cos C=1+\frac{r}{R}$ (A. U. 1951)

44. $4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s}{R}$

45. $(r_1+r_2)\tan\frac{C}{2}=(r_3-r)\cot\frac{C}{2}=c$ (A. U. 1953)

46. $\triangle=\sqrt{r\,r_1\,r_2\,r_3}=r^2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$ (A. U. 1951)

47. $r_1r_2r_3=r^3\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$ (A. U. 1956)

48. (i) $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$ (ii) $(r_2+r_3)\sqrt{\frac{rr_1}{r_2r_3}}=a$

49. $a(rr_1+r_2r_3)=b(rr_2+r_1r_3)=c(rr_3+r_1r_2)$

50. $r_2r_3+r_3r_1+r_1r_2=s^2$ (C. P. 1943)

51. $r_1+r_2+r_3-r=4R$ (A. U. 1955)

52. $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$ (A. U. 1949)

53. $(r_2+r_3)(r_3+r_1)(r_1+r_2)=4Rs^2$

54. $\dfrac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{r_2r_3+r_3r_1+r_1r_2}=4R$ (B. H. U. 1951)

55. $\left(\dfrac{1}{r}-\dfrac{1}{r_1}\right)\left(\dfrac{1}{r}-\dfrac{1}{r_2}\right)\left(\dfrac{1}{r}-\dfrac{1}{r_3}\right)=\dfrac{16R}{r^2(a+b+c)^2}$ (A. U. 1938)

56. $\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}+\dfrac{1}{r_3^2}+\dfrac{1}{r^2}=\dfrac{a^2b^2c^2}{\Delta^2}$ (A. U. 1952)

57. $\left(\dfrac{1}{r}+\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)^2=\dfrac{4}{r}\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)$

58. (i) $r_1+r_2=c\cot\dfrac{C}{2}$ (ii) $\dfrac{rr_1}{r_2r_3}=\tan^2\frac{1}{2}A$ (A. U. 1957)

59. $\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}=\dfrac{1}{2Rr}$ (B.H.U. 1949 '56)

60. $\dfrac{r_1}{bc}+\dfrac{r_2}{ca}+\dfrac{r_3}{ab}=\dfrac{1}{r}-\dfrac{1}{2R}$ (A. U. 1957)

61. $a\cot A+b\cot B+c\cot C=2(R+r)$

62. *If* $R=r_1$, *show that* $\cos A=\cos B+\cos C$

63. *If* $r_1-r=r_2+r_3$, *show that* the triangle is right angled.

64. *If* the diameter of an ex-circle is equal to the perimeter of the triangle, *show that* the triangle in right angled.

65. *If* $\left(1-\dfrac{r_1}{r_2}\right)\left(1-\dfrac{r_1}{r_3}\right)=2$, *show that* the triangle is right angled. (B. H, U. 1954)

66. *If* $8R^2=a^2+b^2+c^2$, *show that* the triangle is right angled. (A. U. 1953)

67. *If* $R=2r$. *show that* the triangle is equilateral. (C. P. 1940)

68. If a, b, c are in A. P., show that r_1, r_2, r_3 are in H. P.

69. The perpendiculars from the angular points of a triangle on the opposite sides meet at O, and $OA=x$, $OB=y$, $OC=z$. Show that $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=\dfrac{abc}{xyz}$.

70. *If* p_1, p_2, p_3 are the perpendiculars from the angular points on the opposite sides of a triangle, *prove that*

(i) $8R^3 = \dfrac{a^2b^2c^2}{p_1p_2p_3}$ (ii) $\dfrac{1}{p_1}+\dfrac{1}{p_2}+\dfrac{1}{p_3}=\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}$

71. If S be the area of the in-circle, and S_1, S_2, S_3, the areas of the escribed circles, then

$$\frac{1}{\sqrt{S}}=\frac{1}{\sqrt{S_1}}+\frac{1}{\sqrt{S_2}}+\frac{1}{\sqrt{S_3}}$$

72. *Prove that* the square of the distance between the circum-centre and the in-centre of any triangle is R^2-2Rr. *If* the circum-centre lies on the in-circle, *prove that*

$$\cos A+\cos B+\cos C=\sqrt{2}$$ (A. U. 1949)

73. In a triangle $a=13$, $b=14$, $c=15$; find r and R.

74. In any triangle, *show that* the area of the in-circle is to the area of the triangle as

$$\pi : \cot \tfrac{1}{2} A \cot \tfrac{1}{2} B \cot \tfrac{1}{2} C$$

12

Solution of triangles

12-1. A triangle has six parts : three sides and three angles. These six elements of a triangle are not independent ; given any three of them, provided one at least is a side, the rest can be found from the relations of the previous chapter, or in other words, the triangle can be *solved.* There are four different ways in which the parts may be given.

Case I : Three sides given.
Case II : Two angles and a side given.
Case III : Two sides and the included angle given.
Case IV : Two sides and an opposite angle given.

12-2. Logarithms

The treatment of logarithm properly belongs to the domain of Algebra and all works on the subject contains it. As it plays, however, an important part in simplifying calculations in the solution of triangles, we shall devote few lines to their chief properties for convenience of reference.

If $x=a^m$, m is called the logarithm of x to the base a ; this is usually written as $m=\log_a x$, or, simply, $m=\log x$, the base remaining understood. So, *the logarithm of a number to a given base is the power to which the base must be raised to give rise to that number.* In practical use the base is usually 10.

12-3. Properties of logarithms

Since the logarithm is an index, and since $a^m=x$ and $m=\log_a x$ are two identical statements, we can derive the properties of logarithms from the relations proved in the Theory of indices in Algebra.

The important properties of logarithms are the following :

(i) $\log_a mnp\cdots = \log_a m + \log_a n + \log_a p + \cdots$

(ii) $\log_a m/n = \log_a m - \log_a n.$

(iii) $\log_a m^r = r \log_a m.$

Expressed in words

(i) The logarithm of a product is equal to the sum of the logarithms of its factors.

(ii) The logarithm of a quotient is equal to the difference of the logarithms of the dividend and the divisor.

(iii) The logarithm of any power of a number is equal to the product of the logarithm of the number into the index of the power.

Cor. $\log_a a = 1$; $\log_a 1 = 0$. That is, logarithm of a number to the same base is unity and the logarithm of unity to any base is zero.

12-4. Characteristic and mantissa

In dealing with logarithms some of which are positive and others negative, it is customary to write them, in all cases, such that the decimal part is positive, the integral part being positive or negative according as the logarithm itself is positive or negative. We give an example. Suppose the logarithm is $-3{\cdot}72388$. This is equal to $-4+1-{\cdot}72388 = -4+{\cdot}27612$. This is written $\bar{4}{\cdot}27612$.

The positive part is called the *mantissa*, and the integral part the *characteristic* of the logarithm. In the above example $\bar{4}$ or -4 is the characteristic and ·27612 the mantissa.

The characteristic of the common logarithm of any number may be found by inspection by the following rules.

(a) the characteristic of the logarithm of a number greater than unity is less by one than the number of digits in its integral part and is positive ; e. g the characteristic of log 643·2 is 2.

(b) the characteristic of the logarithm of a number less than unity is negative and is one more than the number of ciphers (zeros) immediately after the decimal points ; e. g. the characteristic of log ·3476 and log ·0006 are respectively -1 and -5.

The mantissa part of the logarithm is given by the logarithmic Tables. Two numbers having the same significant figures have the same mantissa ; they differ only in characteristics.

12-5. Trigonometrical tables

In solving the triangles, we shall very often take recourse to trigonometrical tables. There are two kinds of such tables. One is *natural* sines, cosines etc. tables in which the values of sines, cosines etc. of angles are given correct to certain places of decimals. The other is *logarithmic* sines, cosines etc. tables in which the logarithms of sines, cosines etc. are shown.

Many trigonometrical ratios are less than unity ; their logarithms are, therefore, negative. To avoid having negative logarithms, they are all increased by 10 and are written L sinA, L cosA etc. They are called Tabular logarithmic sines, cosines etc. Thus

$$L \sin A = 10 + \log \sin A$$
$$L \tan \theta = 10 + \log \tan \theta$$

At the end of the book such tables are given. The natural sines and cosines tables give the values of sines and cosines of all angles between 0° and 90° at intervals of 1′. The sine of an angle is the cosine of the complementary angle. This property is utilised in the arrangement of the table such that the same table serves both as sine table and as cosine table. Sines are given from the left side of the top to the right and downwards and cosines from the right side of the bottom to the left and upwards.

The main part of the table gives sines and cosines of angles at intervals of 10′. The table for differences gives the changes in the sines and cosines for changes in minutes in the angles. Since as the angle increases between 0° and 90°, the sine increases and cosine decreases the changes given in the difference table are to be added in case of sines and subtracted in case of cosines corresponding to an increment in the angle.

Similarly, the table of natural tangents and cotangents gives tangents and cotangents of angles between 0° and 90° at intervals of 1′. The difference is to be added in the case of

the tangent and subtracted in the case of cotangent, if the angle is increased.

The tables of logarithmic sines and logarithmic cosines give the values of L sin θ and L cos θ for values of θ between 0° to 90° at intervals of 1′ (with the help of the difference table). The logarithmic tangent and cotangent tables similarly give the values of L tan θ and L cot θ for all values of θ lying between 0°—90° at intervals of 1′. When the angle gets an increment the difference is to be added in case of sines and tangents and subtracted in case of cosines and cotangents.

12-6. Principle of proportional parts

Sometimes it is required to find the trigonometrical ratios or their logarithms of an angle not exactly given in the table, e. g. 34° 23′ 20″. Tables give 34° 23′ and 34° 24′ only.

To meet such cases we use the *Principle of proportional parts* which may, without proof, be stated as follows :

The change in the value of a function of a variable is, in most cases, approximately proportional to the change in the variable, provided the change in the variable is very small.

Sin θ, tan θ, L sin θ etc. are all functions of the variable θ.

12-7. We shall now consider one by one the different cases that may arise in solving the triangle.

Case I. Having given the three sides a, b, c

The angles are usually determined from the formulæ of the type

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}.\ \text{etc.}$$

From above, $L\tan\frac{A}{2}=10+\frac{1}{2}\log\frac{(s-b)\,(s-c)}{s\,(s-a)}.$

Thus $\frac{A}{2}$ is known from Tables and hence A. Similarly B can be found. C is obtained from the relation $C=180°-A-B$.

We could have solved the triangle from the formulae of the type.

$$\cos\frac{A}{2}=\sqrt{\frac{s\,(s-a)}{bc}}\ \text{etc.}$$

$$\text{or,}\quad \sin\frac{A}{2}=\sqrt{\frac{(s-b)\,(s-c)}{bc}}\ \text{etc.}$$

taking L cos $\frac{A}{2}$ or L sin $\frac{A}{2}$ and using the tables. But in the tangent formulæ we require only *four* logarithms, s, $s-a$, $s-b$, $s-c$, to solve the triangle, whereas to solve from sine or cosine formulæ we require *six* logarithms, s, $s-a$, $s-b$, a, b, c. Hence tangent formulæ are the best.

We may also use the cosine formulæ of the type :

$$\cos A=\frac{b^2+c^2-a^2}{2bc} \text{ etc.}$$

These formulæ, though uniquely give the angle, are not suitable for logarthmic calculation. Also, when an angle is to be determined by using an approximate table, logarithmic tangent table gives the best result. This can be proved mathematically.

Cass II Having given two angles and a side

Let B, C, and a be given.

$$\therefore \quad A=180°-B-C$$

The unknown sides are obtained from the formulæ

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\therefore \quad b=a\frac{\sin B}{\sin A}; \quad c=a\frac{\sin C}{\sin A}$$

or, taking logarithms,

$$\log b=\log a+L\sin B-L\sin A$$
$$\log c=\log a+L\sin C-L\sin A$$

Case III Given two sides and the included angle

Let b, c and A be given.

B and C may be determined from

$$\left.\begin{aligned}\tan\frac{B-C}{2}&=\frac{b-c}{b+c}\cot\frac{A}{2}\\ B+C&=180°-A\end{aligned}\right\} \text{ and}$$

Taking logarithms, we have

$$L\tan\frac{B-C}{2}=\log(b-c)-\log(b+c)+L\cot\frac{A}{2}$$

Having found B and C, a is known from the relation

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

i.e. $\log a = \log b + L\sin A - L\sin B$

or, $\quad = \log c + L\sin A - L\sin C$

Case IV Having given two sides and opposite angle

Let a, b, A be given.

From the relation $\sin B = \frac{b}{a}\sin A$, B is to be found.

Three different cases may arise

(i) *if* $a < b\sin A$, then $\frac{b}{a}\sin A > 1$, so that $\sin B > 1$, which is impossible ; *there is no solution.*

(ii) *if* $a = b\sin A$, then $\frac{b}{a}\sin A = 1$, so that $\sin B = 1$ and *B has the only value 90°.*

(iii) *if* $a > b\sin A$, then $\frac{b}{a}\sin A < 1$, and *there are two values of B*, one *acute* and the other *obtuse*.

(α) *if* $a < b$, then $A < B$; so B may be acute or obtuse and *both values of B are admissible.* This is called the *Ambiguous case.*

(β) *if* $a = b$, then $A = B$ and if $a > b$, then $A > B$; in each case B cannot be obtuse ; so the *smaller value of B is only* admissible.

When B is found, C is known from $C = 180° - B - A$.

Finally, c is found from, $\quad c = \frac{a}{\sin A}\sin C$

Note : The only case where an ambiguous solution arises is that where the smaller of the given sides is opposite to the given angle.

12-8. Geometrical discussion of the ambiguous case

Take a line AX unlimited towards X. Let $\angle XAC$ be A and $AC = b$. Draw CD perp. to AX ; so $CD = b\sin A$. With centre C and radius a draw a circle.

(i) if $a<b \sin A$, the circle will not meet AX ; no triangle is possible with the given elements.

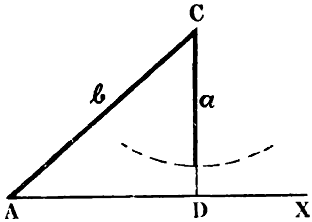

Fig. 36

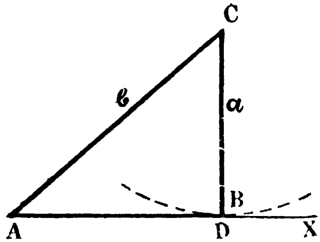

Fig. 37

(ii) if $a=b \sin A$, the circle touches AX at D ; a rt. angled triangle is the possible solution.

(iii) if $a>b \sin A$, the circle will cut AX in two points B_1, B_2.

(α) These two points will be on the same side of A when $a<b$, in which case there are two possible $\triangle$s AB_1C and AB_2C. **This is the ambiguous case.**

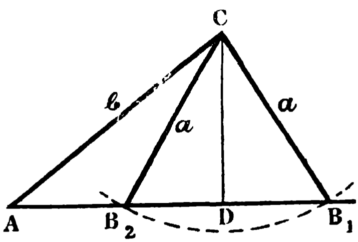

Fig. 38

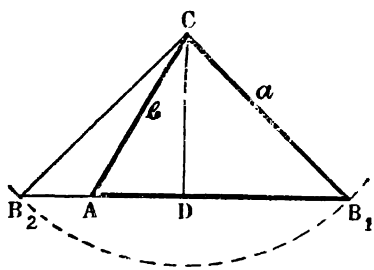

Fig. 39

(β) The points B_1, B_2 will be on the opposite sides of A when $a>b$. In this case there is only one solution. Since $\angle CAB_2$ is $180°-A$, $\triangle AB_2C$ does not satisfy the given data.

(γ) if $a=b$, B_2 coincides with A and there is only one solution.

12-6. Three angles of a triangle given

When the three angles of a triangle are only given, the triangle cannot be solved. There are infinite number of equiangular triangles all satisfying the given data.

The relation $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ only enable us to find the ratios of the sides but not their actual lengths.

12-7. Illustrated examples

Ex. 1. *The sides of a triangle are 5, 7, 9. Find the greatest angle ; given log 2=·3010300, log 3=·4771213, L cos 47°53′= 9·8264910 : diff. for 60″=1397.*

Let $a=5, b=7, c=9$; so C is the greatest angle and

$$\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$$

$$=\sqrt{\frac{21}{2}\times\frac{3}{2}\times\frac{1}{5\times 7}} \quad \left(\because \ s=\frac{5+7+9}{2}=\frac{21}{2}\right)$$

$$=\sqrt{\frac{9}{20}}$$

$$\therefore \ L\cos\frac{C}{2}=10+\tfrac{1}{2}(2\log 3-\log 2-1)$$

$$=10+\tfrac{1}{2}(\cdot 9542426-\cdot 3010300-1)$$

$$=10+\tfrac{1}{2}\times\bar{1}\cdot 6532126$$

$$=10+\bar{1}\cdot 8266063$$

$$=9\cdot 8266063$$

But $\quad L\cos 47°53'=9\cdot 8264910.$

Diff : $\quad 1153$

$$\therefore \ \text{Proportional decrease}=\frac{1153}{1397}\times 60''=49\cdot 5''$$

$$\therefore \ \frac{C}{2}=47°52'10\cdot 5''$$

Ex. 2. *The sides of a triangle are 323, 481 and 642. Find all the angles. Use mathematical tables.*

Let $a=323$, $b=481$ and $c=642$

$$\therefore \quad s=\frac{323+481+642}{2}=723$$

$$\therefore \quad \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{242\times 81}{723\times 400}}$$

$$\therefore \quad L\tan\frac{A}{2}=10+\tfrac{1}{2}\ (\log 242+\log 81-\log 723-\log 400)$$

$$=10+\tfrac{1}{2}(2{\cdot}38382+1{\cdot}90849-2{\cdot}85914-2{\cdot}60206)$$
$$=9{\cdot}415555$$
$$=9{\cdot}41556 \text{ (upto 5 decimals)}$$

Now, $\quad L\tan 14°36'=9{\cdot}41579$ $\quad$ (from table)

and $\quad L\tan 14°35'=9{\cdot}41527$

$\therefore \quad$ diff. for $60''=\quad 52$

$$\because \quad L\tan\frac{A}{2}=9{\cdot}41556$$

and $\quad L\tan 14°35'=9{\cdot}41527$

$\therefore \quad$ diff. $=\quad 29$

$$\therefore \quad \text{Proportional increase}=\frac{29}{52}\times 60''=33{\cdot}5''$$

$$\therefore \quad \frac{A}{2}=14°35'33{\cdot}5''. \quad \boxed{\therefore \quad A=29°11'7''}$$

$$\text{Again,} \quad \tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}=\sqrt{\frac{81\times 400}{723\times 242}}$$

$$\therefore \quad L\tan\frac{B}{2}=10+\tfrac{1}{2}\ (\log 81+\log 400-\log 723-\log 242)$$

$$=10+\tfrac{1}{2}(1{\cdot}90849+2{\cdot}60206-2{\cdot}85914-2{\cdot}38382)$$
$$=9{\cdot}63380$$

But, $\quad L\tan 23°18'=9{\cdot}63377$

Diff$=\quad 3$

Also, from table, diff. for $60''=35$

$\therefore$ Proportional increase $=\frac{3}{35}\times 60''=5{\cdot}2''$

$\therefore$ $\frac{B}{2}=23°18'5{\cdot}2''$. $\boxed{\therefore\ B=46°36'10{\cdot}4''}$

$$\therefore\ C=180°-(A+B)$$

or, $\boxed{C=104°42'72{\cdot}6''}$

Ex. 3. *In a triangle ABC, $B=60°$, $A=38°20'$ and $b=64$ feet. Find a*

We have $C=180°-(60°+38°20')=81°40'$

Now, $$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\therefore\ a=b.\frac{\sin A}{\sin B}=64.\frac{2}{\sqrt{3}}.\sin 81°40'$$

$$\therefore\ \log a=7\log 2-\tfrac{1}{2}\log 3+L\sin 81°40'-10$$
$$=7\times{\cdot}30103-\tfrac{1}{2}\times{\cdot}47712+9{\cdot}99539-10$$
$$=2{\cdot}10721-{\cdot}23856+9{\cdot}99539-10$$
$$=1{\cdot}86404$$

From the antilogarithm table, $a=73.12$ feet

Ex. 4. *Given $b=5$, $c=3$ and $A=120°$, find B and C.*

(C. U. 1949)

We have $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$=\frac{5-3}{5+3}\cot 60°$$

$$=\frac{1}{4}.\frac{1}{\sqrt{3}}$$

$$=\frac{1}{\sqrt{48}}$$

$$\therefore\ L\tan\frac{B-C}{2}=10+\log 1-\frac{1}{2}\log 48$$

$$=9{\cdot}1593794$$

From the table of logarithmic tangents.

$L \tan 8°12' = 9{\cdot}1586706$, and

diff. for $1' = 8940$

Here, $L \tan \frac{B-C}{2} = 9{\cdot}1593794$

and $L \tan 8°12' = 9{\cdot}1586706$

$\quad$ diff$=$ $\quad 7088$

$\therefore$ Proportional increase$=\frac{7088}{8940} \times 60'' = 48''$

$$\therefore \quad \frac{B+C}{2} = 8°\ 12'\ 48'' \qquad \ldots\ldots(i)$$

But, $\frac{B+C}{2} = 90° - \frac{A}{2}$

$= 90° - 60°$

$= 30° \qquad$ (ii)

$\therefore$ From (i) and (ii) by addition and subtraction,

$B = 38°\ 12'\ 48''$

$C = 21°\ 47'\ 12''$

Ex. 5. *If $b = 63$, $c = 36$, $C = 29°\ 23'\ 15''$, find B;*
Given $\log 2 = {\cdot}3010300$, $\log 7 = {\cdot}8450980$

$L \sin 29°23' = 9{\cdot}6907721$, diff. for $1' = 2243$

$L \sin 59°10' = 9{\cdot}9338000$, diff. for $1' = 755$

Is the case ambiguous?

$\sin B = \frac{b}{c} \sin C = \frac{63}{36} \sin C$

$= \frac{7}{4} \sin 29°\ 23'\ 15''$

$\therefore \quad L \sin B = \log 7 - 2 \log 2 + L \sin 29°\ 23'\ 15''$

$= {\cdot}8450980 - {\cdot}6020600 + 9{\cdot}6908282$

($\because$ diff. for $60'' = 2243$)

$= 9{\cdot}9338662$

But $L \sin 59°10' = 9{\cdot}9338222$

$\quad$ diff.$=$ $\quad 440$

$\therefore$ proportional increase$=\frac{440}{755} \times 60'' = 35''$

$\therefore \quad B = 59°\ 10'\ 35''$

Also, since $c<b$, there is another value of B, supplementary to the above, i.e. $B=120°49'25''$. Hence the case is ambiguous.

Ex. 6. *The angles of a triangle are in the ratio 2 : 3 : 7 ; prove that the sides are in the ratio* $\sqrt{2} : 2 : \sqrt{3}+1$.

$$A : B : C = 2 : 3 : 7$$

Also, $A+B+C=180°$

$$\therefore \quad A=30°, \; B=45°, \; C=105°$$

$$\therefore \quad a : b : c = \sin A : \sin B : \sin C$$

$$= \sin 30° : \sin 45° : \sin 105°$$

$$= \frac{1}{2} : \frac{1}{\sqrt{2}} : \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \sqrt{2} : 2 : \sqrt{3}+1$$

Examples 12

1. The sides of a triangle are 5, 8, 11 ; find the greatest angle ; given log 7=·8450980, L sin 56°47′=9·9225205, L sin 56° 48′=9·9226032

2. The sides of a triangle are 4, 5, 6. Find B, having given, log 2=·3010300, L cos 27°53′=9·9464040 diff. for 1′ =·0000669

3. The sides of a triangle are 2, 3, 4. Find the greatest angle. Given log 2=·30103, log 3=·4771213, L tan 52°14′ =10·1108395, L tan 52°15′=10·1111004

4. The sides of a triangle are 9, 10, 11 ; find the angle opposite to the side 10, given log 2=·30103 and L tan 29°30′ =9·7526420, L tan 29°29′=9·7523472 (C. U. 1934)

5. The sides of a triangle are 15, 19, 24 ; find the greatest angle of the triangle. Given log 5·7=·75587. L cos 88°59′ =8·24903 ; diff. for 1′=718 (C. U. 1936)

6. The sides of a triangle are 7, 8, 9; solve the triangle using mathematical tables. (C. U. 1938)

7. If $a=74$, $b=26$, $c=60$, find the value of the angle A, having given log 9 = ·9542425. log 13 = 1·1139434, L sin 56°19′ = 9·9201896 and diff. for 1′ = 842

8. If $b=1000$, A = 45°, C = 68°17′40″, find the least side, having given log 2 = ·3010300, log 7·6986 = ·8864118, diff. for 1 = 57, L sin 66°42′ = 9·9630538, diff. for 1′ = 544

9. In a triangle ABC, B = 60°, C = 65°26′ and $b=162$ Find a. Use tables.

10. In a triangle ABC, A = 38°20′, B = 45° and $b=64$ feet. Find c having given log 2 = ·30103, L sin 83°20′ = 9·99705 and log ·089896 = $\bar{2}$·95374

11. If B = 45°, C = 10° and $a=200$ ft, find b, having given log 2 = ·30103, Lsin 55° = 9·9133645, log 1726·4 = 3·2371414 log 1726·5 = 3·2371666 (C. U. 1947)

12. If in a triangle ABC, $a=19$, A = 52°28′ and C = 93°40′, find b, having given log 19 = 1·2787536,
log 27037 = 4·4319585, log 27038 = 4·4319746
L sin 52°28′ = 9·8992727, L sin 33°52′ = 9·7460595

13. If A = 44°, C = 70°, $b=1006·62$, find a and c; given
L sin 44° = 9·8417713, log 100662 = 5 0028656
L sin 66° = 9·9607302, log 103543 = 5·0151212
L sin 70° = 9·9729858, log 7654321 = 6·8839067

14. In a triangle $b=2·25$, $c=1·75$, A = 54°, find B and C, having given, log 2 = ·301030, L tan 63° = 10·292834, L tan 13°47′ = 9·389724, L tan 13°48′ = 9·390270 (C.U. 1931)

15. Given $b=643$, $c=365$, and A = 144°48′, find B and C using tables

16. If the sides a and b are in the ratio $7:3$ and the included angle $C=60°$, find A and B ; given $\log 2=\cdot 3010300$, $\log 3=\cdot 4771213$, $L \tan 34°42'=9\cdot 8403776$, diff. for $1'=2699$

17. If $a=681$, $c=243$, $B=50°42$, solve the triangle by the use of tables.

18. Two sides of a triangle are 14 and 11 and the included angle is 60°. Find the remaining angles having given $L \tan 11°44'=9\cdot 3174299$, $L \tan 11°45'=9\cdot 3180640$ (A. U. 1955)

19. Two sides of a triangle are 80 and 100 ft. and the included angle is 60°. Find the other angles. (C. U. 1946)

20. Given $a=70$, $b=35$, $C=36°52'12''$, $\log 3=\cdot 4771213$, $L \cot 18°26'6''=10\cdot 4771213$. Calculate A and B. (C. U. 1935, 37)

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21. In a triangle $b=324$, $c=562$, and $B=32°20'$, find A, C and a ; use tables.

22. In a triangle $a=73\cdot 4$, $b=64\cdot 9$, and $B=48°13'25''$; find A, having given

$$\log 734=2\cdot 8656961$$
$$\log 649=2\cdot 8122447$$
$$L \sin 48°13'25''=9\cdot 8725936$$
$$L \sin 57°30' =9\cdot 9260292 \text{ (diff. for } 1'=804)$$

Is the case ambiguous ?

23. If $a=145$, $b=178$, $B=41°40'$, find A ; given
$\log 178=2\cdot 2504200$, $L \sin 41°10'=9\cdot 8183919$,
$\log 145=2\cdot 1613680$, $L \sin 32°45'35''=9\cdot 7293399$

24. Supposing the data for the solution of a triangle to be as in the three following cases, point out whether the solution is ambiguous or not.

(i) $A=30°$ $a=125$ feet, $c=250$ feet.
(ii) $A=30°$, $a=200$ feet, $c=250$ feet.
(iii) $A=30°$, $a=200$ feet, $c=125$ feet.

Given $\log 2=\cdot 30103$
$\log 6\cdot 0389=\cdot 7809578$, $L \sin 38°41'=9\cdot 7958800$,
$\log 6\cdot 0390=\cdot 7809650$, $L \sin 8°41'=9\cdot 1789001$

25. In the ambiguous case, given a, b and A, prove that the difference between the two values of c is $2\sqrt{a^2-b^2\sin^2 A}$.

(A. U. 1952)

26. If a, b and A are given and if c_1 and c_2 are the values of the third side in the ambiguous case, show that if $c_1 > c_2$

(i) $c_1 - c_2 = 2a \cos B$ (B. H. U. 1928)

(ii) $c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = 4a^2\cos^2 A$

(B. H. U. 1935, P. U. 1936)

(iii) $a \cos\dfrac{C_1 - C_2}{2} = b \sin A$ (A. U. 1941)

27. Show that in the case that admits of two solutions, the two values of C satisfy the equation

$$\frac{(a-b)^2}{1+\cos C} + \frac{(b-a)^2}{1-\cos C} = \frac{2a^2}{\sin A}.$$ (B. H. U. 1942)

13

Graphs of trigonometrical functions

13-1. Variations in the sine of an angle as the angle continuously increases

Let the line OP, of length r, start from the position OA and revolving continuously about O counter-clockwise trace out the circle of radius r. From any pt. P on the circumference drop PM perp. to OA or OA produced. If θ be the angle described by OP, we have,

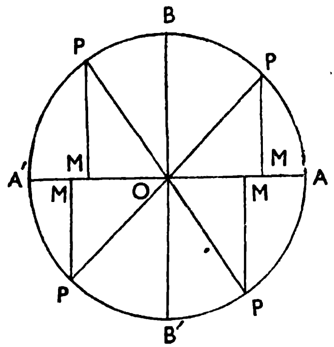


Fig. 40

$$\sin\theta = \frac{MP}{OP} = \frac{MP}{r}$$

As P revolves continuously in the anti-clockwise direction, MP *increases* while P moves from A to B ; *decreases* from B to A′, *numerically increases* from A′ to B′, but is negative, then *decreases numerically*, remaining negative, from B′ to A. Since r does not change in sign or magnitude, we have only to consider the changes in MP as P moves round the circle. The greatest values of MP are OB and OB′, and are equal to r. the least value is zero when P coincides with A and A′.

Hence we have the following results

In the first quadrant, sin θ is positive and increases from 0 to 1

In the second, sin θ is positive and decreases from 1 to 0

In the third, sin θ is negative and decreases from 0 to—1

In the fonrth, sin θ is negative and increases from—1 to 0

When OP has described 2π by completing one revolution, it is again at OA and continues to retrace the path. So the changes in $\sin\theta$ when θ lies between 2π and $2\frac{1}{2}\pi$ are the same as when θ is between 0 and $\frac{1}{2}\pi$; again its changes between $2\frac{1}{2}\pi$ and 3π are the same as between $\frac{1}{2}\pi$ and π, and so on. The same cyclical changes repeat indefinitely every time θ increases by 2π.

This fact is expressed by saying that $\sin\theta$ is a **periodic function** of θ, its **period** being 2π.

Note : We may similarly trace the variations of the other trigonometrical ratios, and all of them are periodic functions having the same period 2π, after each of which the same cycle of values is repeated.

We give below two diagrams which exhibit the changes in cosine and tangent ratios.

	$\cos 90° = 0$	
	cos *negative* *and increasing*	cos *positive* *and decreasing*
$\cos 180° = 1$		$\cos 0° = 1$
	cos *negative* *and decreasing*	cos *positive* *and increasing*
	$\cos 270° = 0$	

	$\tan 90° = 0$	
	tan *negative* *and decreasing*	tan *positive* *and increasing*
$\tan 180° = 0$		$\tan 180° = 0$
	tan *positive* *and increasing*	tan *negative* *and decreasing*
	$\tan 270° = \infty$	

13-2. Graphical representation

The trigonometrical functions ($\sin x$, $\cos x$ etc.), like the algebraical functions, may be represented graphically, the method of drawing the graph being the same as in Algebra.

The angles are represented along the X—axis and the values of the trigonometrical functions corresponding to the angle along Y—axis. Both the representations are on a suitable scale. We thus get a series of points. Joining them free-hand, we get the graph.

13-3. Sine-graph

Let $y = \sin x$; from the natural sine-table the values of x differing by 10° are found.

x	−90°	−80°	−70°	−60°	−50°	−40°	−30°	−20°	−10°	0°
y=sin x	−1	−·98	−·94	−·87	−·77	−·64	−·50	−·34	−·17	0

x	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	etc.
y	·17	·34	·50	·64	·77	·87	·94	·98	1	·98	·94	etc.

Let 1 small division along OX represent 10° and 10 small divisions along OY represent unity. The points given by the

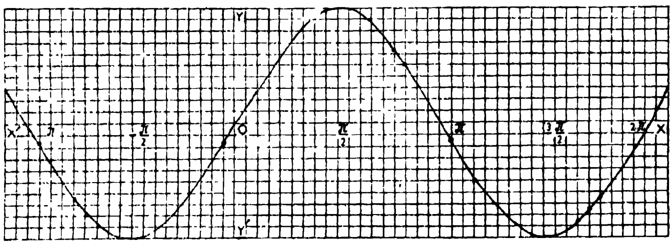


Fig. 41—Sine graph

above tabulated values are plotted and joined free-hand. We then obtain the graph.

Note : The sine table gives the sines of angles from 0°—90° only, for other angles, we have the formulæ $\sin(-\theta)=-\sin\theta$, $\sin(180\mp\theta)=\pm\sin\theta$,

Peculiarities of sine graph

(i) The graph is continuous and wavy in form.

(ii) The greatest and least values of $\sin x$ are $+1$ and -1, the maximum and minimum values being attained for values of x which are odd multiples of $\pi/2$.

(iii) Since $\sin(2n\pi+x)=\sin x$, the portion of the curve between 0°—360° is repeated endlessly after every multiple of 360° towards both the positive and negative values of x. Sin x is thus called a periodic function of period 4 right angles.

(iv) The curve between 0°—180° is symmetrical about the ordinate at 90°.

(v) The curve between 180°—360° is similar to that between 0°—180° but lies below the X—axis.

13-4. Cosine-graph

Let $y=\cos x$; from the natural cosine table the values of y corresponding to values of x differing by 10° are found.

Let 1 small division along OX represent 10° and 10 small divisions along OY represent unity. The points given by the above tabulated values are plotted and joined free-hand. We then obtain the graph.

x	−90°	−80°	−70°	−60°	−50°	−40°	−30°	−20°	−10°	0°
y=cos x	0	·17	·34	·50	·64	·77	87	·94	·98	1

x	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	etc.
y	·98	·94	·87	·77	·64	·50	·34	·17	0	−·17	−·34	etc.

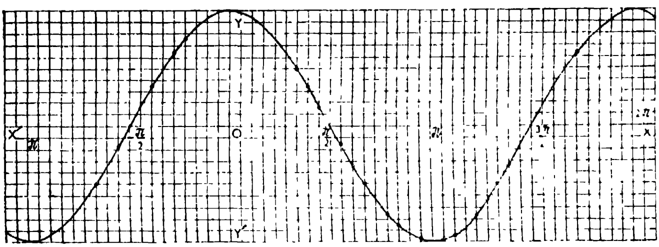


Fig. 42—Cosine graph

Peculiarities of cosine graph

(i) The graph of cos x is the same as the sine graph moved to the left through a space corresponding to 90° ; this is because sin $(90°+x)=\cos x$.

(ii) The graph is continuous and periodically repeated after every interval of 360°.

(iii) The maximum and minimum values of cos x is $+1$ and -1, occuring at 0° and even multiples of 90°.

13-5. Tangent-graph

Let $y=\tan x$; from the natural tangent table the values of y corresponding to values of x differing by 10° are found.

Let 1 small division along OX represent 10° and 3 small divisions along OY represent unity. The points given by the

above tabulated values are plotted and joined free-hand. We then obtain the graph.

x	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$y=\tan x$	0	·18	·36	·58	·84	1·19	1·73	2·75	5·67	∞

x	100°	110°	120°	130°	140°	150°	160°	170°	180°	etc.
y=tan x	−5·67	−2·75	−1·73	−1·19	−·84	−·58	−·36	−·18	0	etc.

Fig. 43—Tangent graph

Peculiarities of tangent graph

(i) The curve is not continuous, but consists of separate branches ; the discontinuities occur at the values of x corresponding to odd multiples of 90°.

(ii) The curve between 0°—90° lies above the X—axis and that between 90°—180° below the X—axis.

(iii) As the angle approaches 90°, the tangent increases very rapidly remaining positive throughout and is $+\infty$ at

$90°$. As the angle passes through $90°$, the tangent becomes $-\infty$. As the angle changes from $90°-180°$, the values of the tangent are those already traced but in the reverse order.

(iv) The lines parallel to Y—axis corresponding to odd multiples of $\pi/2$ are continuously approached by the graph on either side but never exactly met. Such lines are called *asymptotes* to the curve.

(v) The curve is repeated after every interval of $180°$.

13·6. Graphs of cot x, cosec x, sec x

The graphs of cot x, sec x and cosec x are left as exercises to the students.

The cotangent graph is similar to the tangent-graph shifted through $90°$ towards the right or left. It is discontinuous at $0°$ and at multiples of π. The graph is repeated after every interval of $180°$.

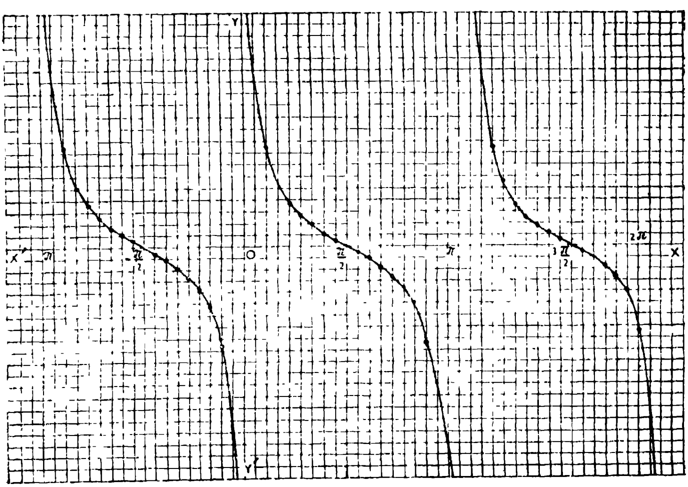


Fig. 44—Cotangent graph

The cosecant graph consists of separate branches. It is discontinuous at $x=0$ and $x=n\pi$. The lines given by $x=n\pi$ are asymptotes to the curve. The curve between $0°-180°$ lies

above the X—axis and that between 180°—360° above the X—axis. The same curve is repeated after every multiple

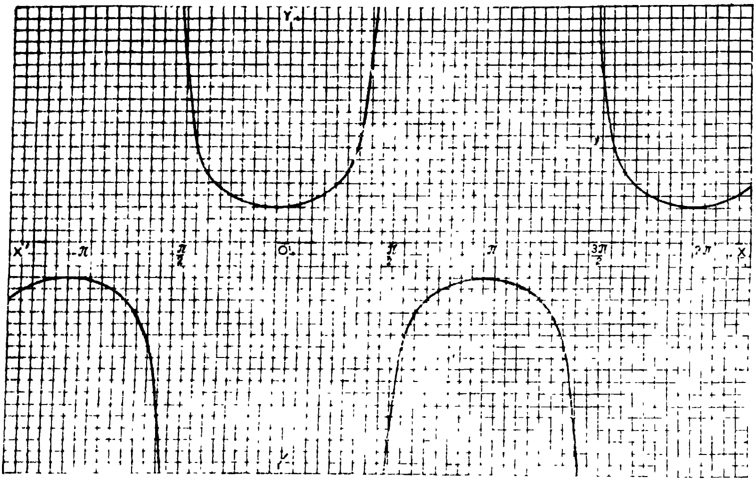


Fig. 45—Secant graph

of 360°. The secant-graph is similar to the cosecant graph shifted to the left through a space corresponding to 90°. As illustratation, therefore, we have given only the secant graph.

13-7. Graphs of other trigonometrical functions

The graphs of other trigonometrical expressions may be drawn in a similar manner. Sometimes, instead of proceeding with the given expressions directly, it is advisable to reduce them, if possible, to simpler forms. The worked out examples will illustrate the method.

13-8. Graphical solution of equations

Trigonometrical equations, just like the algebraic equations, may be solved graphically. The method is illustrated by worked out examples.

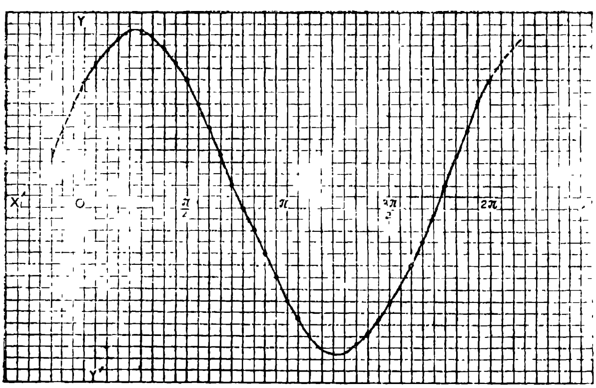
13-9. Illustrated examples

Ex. 1. *Draw the graph of $y=\sin x+\cos x$ between the range $x=0$ to $x=2\pi$.* (C. U. 1934, P. U. 1943)

Let $y=\sin x+\cos x$; from the natural sine table and cosine table, by addition, the values of y corresponding to the values of x differing by 15° are found.

x	0	15°	30	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
sin x	0	·26	·5	·7	·87	·97	1	·97	·87	·7	·5	·26	0
cos x	1	·97	·87	·7	·5	·26	0	−·26	−·5	−·7	−·87	−·97	−1
y	1	1·23	1·37	1·4	1·37	1·23	1	·71	·37	0	−·37	−·71	−1

x	195°	210°	225°	240°	255°	270°	285°	300°	315	330°	345°	360°
sin x	−·26	−·5	−·7	−·87	−·97	−1	−·97	−·87	−·7	−·5	−·26	0
cos x	−·97	−·87	−·7	−·5	−·26	0	·26	·5	·7	·87	·97	1
y	−1·23	−1·37	−1·4	−1·37	−1·23	−1	−·71	−·37	0	·37	·71	1



Let 2 small divisions along OX represent 15° and 10 small divisions along OY represent unity.

The points given by the above tabulated values are plotted and joined free-hand. We then obtain the graph.

Ex. 2. *From the graph of Ex. 1, find the values of x for which $y=0$, $y=$maximum and $y=$minimum.* (C. U. 1938)

From the above graph

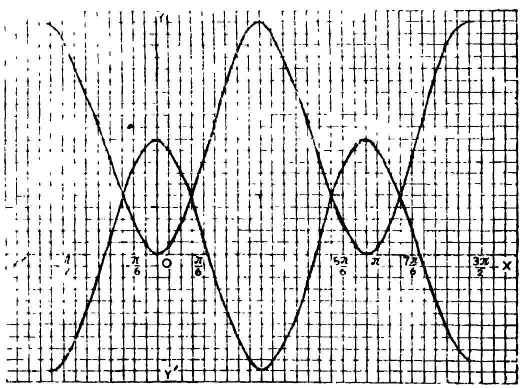
$$y=0 \text{ when } x=\frac{\pi}{4} \text{ and } \frac{7\pi}{4}$$

$$y \text{ maximum when } x=\frac{\pi}{4}$$

$$y \text{ minimum when } x=\frac{5\pi}{4}.$$

Ex. 3. *Solve graphically, $2\sin^2 x=\cos 2x$, giving solutions of x lying between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$.* (C. U. 1936, '46, '48)

Draw the graph of $y=2\sin^2 x=(1-\cos 2x)$ and also, separately, $y=\cos 2x$, making use of natural cosine table between



the range $x=-\frac{\pi}{2}$ to $x=\frac{3\pi}{2}$ at intervals of $\frac{\pi}{12}$, say. The units chosen should be the same in drawing the two separate graphs.

We see that the graphs intersect and thus have the same abscissae and ordinates at the points for which

$$x=-\frac{\pi}{6},\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6},$$

which are the required solutions in the given range.

Examples 13

Solve graphically the following equations

1. $\cos 2x = \sin x$, between 0 to $\frac{\pi}{2}$.

2. $\tan 2x = \tan x$ between 0 to $\frac{\pi}{2}$.

3. $\cos x = \frac{1}{2}$ between $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

4. $\cos x = x$ between 0 to $\frac{\pi}{2}$.

5. $\cos x = \sqrt{3} \sin x$ between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

6. $\sin x = \sin 2x$ between 0 to 2π.

7. $\cot \theta - \tan \theta = 2$, between 0 to π.

8. Draw the graphs of $\sin \theta$ and $\cos \theta$ between $\theta = 0$ and $\theta = \pi$. Find the points where the graphs intersect. (C.U. 1946)

9. Construct the graphs of $\tan x$ and $\cos x$ between 0 and $\pi/2$ for x, making a tabulation of the values of y dividing the interval into 9 equal parts.

If $\tan x = \cos x$, find approximately the values of x from the above two graphs. (C. U. 1943)

10. Obtain graphically a solution of the equation $\tan x = 1$ between $x = 0$ and $x = \pi/2$ (C. U. 1937)

11. Solve graphically $x - \tan x = 0$ between 0 to $\frac{\pi}{2}$.

(C. U. 1945, '54)

12. Solve graphically $5 \sin \theta + 2 \cos \theta = 5$ between 0 to $\frac{3\pi}{2}$.

(C. U. 1947)

13. Solve graphically $\cot \theta - \tan \theta = 2$ between $\theta = 0$ and $\theta = \pi$. (C. U. 1949)

14. Trace the changes in the sign and magnitude of $\cos A - \sin A$ as A changes from 0 to 2π. (A. U. 1951)

15. Draw the graph of $y = \sin x$ from $x = 0$ to $x = \pi$ and from the graph find the angles whose sine is 0·7.

(P. U. 1940, 45)

16. Solve graphically the equation $\tan x = \cot x$, between $x=0$ and $x=\pi/2$. (C. U. 1956)

17. Draw the graphs of $y=\cos 2x$ and $y=2x-1$ and hence solve the equation $x=\cos^2 x$. (C. U. 1955)

18. Draw the graph of $3 \sin x + 4 \cos x$. What is its maximum value ? (C. U. 1950)

19. Sketch the graphs of $y=x$, $y=\sin x$ and $y=\tan x$ in the range between $-\pi/2$ and $\pi/2$ with reference to the same axes in x and y. From the nature of the graphs near the origin can you suggest any relation among them at the origin ? (C.U. 1952)

14

Easy problems of heights and distances

14-1. Trigonometry is an applied branch of mathematics and we shall now show how the results of trigonometry may be applied to determine the heights and distances of objects when they cannot be directly measured. In doing this we are to measure certain important angles and surveying instruments like the *sextant* and the *theodolite* are used for this purpose.

14-2. Angle of elevation. Angle of depression

The angle of inclination of the line joining the observer's eye to the object, to the horizontal is called the ***angle of elevation*** (or simply *elevation*) of the object if the object is ***above*** the observer. The same angle is said to be the ***angle of depression*** (or simply *depression*) if the object is *below* the observer.

Let P be an object, O the eye of the observer and OH, the

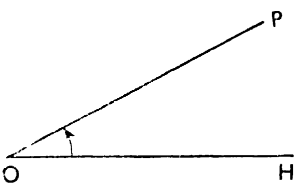


Fig. 46

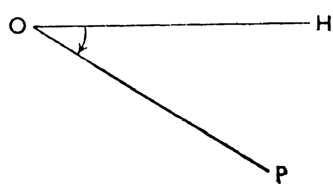


Fig. 47

horizontal through O. Then the angle POH, in the first figure, is the elevation (or altitude) and in the second the depression.

14-3. Illustrated examples

The method of calculation will be best understood from the following worked out examples.

Ex. 1. *A flagstaff stands on a horizontal plane. From a distance 40 ft. from its foot on the ground, its angle of elevation is 60°; calculate the height of the flagstaff.*

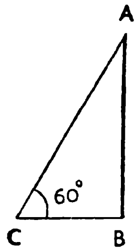


Fig. 48

Let AB be the flagstaff; C is the point of observation.

$$BC = 40\text{ft}.$$

It is required to find the length of AB.

$$\begin{aligned} \text{Now,}\quad AB &= BC \tan 60° \\ &= 40.\ \sqrt{3} \\ &= 40 \times 1{\cdot}732 \\ &= 69{\cdot}28\text{ft}. \end{aligned}$$

So the height is 69·28 ft.

Ex. 2. *The angle of elevation of the top of a tower is 30°; on walking 100 yds. nearer the elevation is found to be 60°. Find the height of the tower.*

Let PQ be the tower of height h.

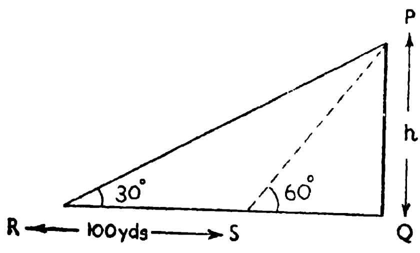


Fig. 49

$\angle PRQ = 30°$, $\angle PSQ = 60°$.

$\therefore\quad RS = 100$ yds.

Now, $\dfrac{h}{RQ} = \tan 30°$ and $\dfrac{h}{SQ} = \tan 60°$.

$$\begin{aligned} \therefore\quad RQ &= h \cot 30° \\ SQ &= h \cot 60° \end{aligned}$$

$$\therefore\quad RQ - SQ = h\ (\cot 30° - \cot 60°)$$

or, $100 = h\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)$

$= \frac{h(3-1)}{\sqrt{3}}$

$\therefore \quad h = \frac{100\sqrt{3}}{2} = 50\sqrt{3} = 50 \times 1{\cdot}732$

$= 86{\cdot}60$ yds.

Ex. 3. *A pole is situated on the top of a tower 50 feet high. The elevation of the top of the tower to an observer is 30°, the elevation of the top of the pole is 45°. Find the length of the pole and the distance of the observer from the foot of the tower.*

Let x be the length of the pole BD, BC be the tower. A is the point of observation,

$\therefore \quad \angle DAC = 45°,$

$\angle BAC = 30°.$

$BC = 50$ft.

Now, $\frac{x+50}{AC} = \tan 45° = 1$

$\therefore \quad x + 50 = AC.$

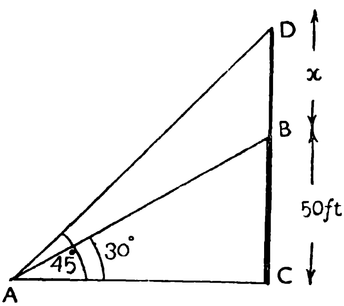


Fig. 50

and $\frac{50}{AC} = \tan 30° = 1/\sqrt{3} \quad \therefore \quad 50\sqrt{3} = AC.$

$\therefore \quad x + 50 = 50\sqrt{3}$

$x = 50\sqrt{3} - 50$

$= 36{\cdot}6$ ft. (approx)

and $y = 50\sqrt{3} = 86{\cdot}6$ ft. (approx)

Ex. 4. *The shadow of a tower standing on a level plane is found to be 40 feet longer when the sun's altitude is 45° than when*

it is 60°. Find the height of the tower correct up to two decimal places ($\sqrt{3}=1{\cdot}7320$) (C. U. 1957)

Let h be the height of the tower AB. AD is the ray of the sun of altitude 45°. BD is the length of the shadow of the tower.

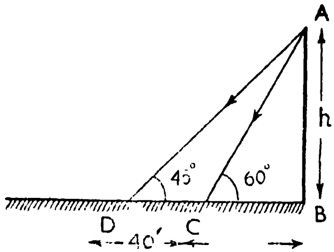


Fig. 51

AC is the ray of the sun when its altitude is 60°.

BC is now the length of the shadow of the tower. By the problem, DC=40 ft.

Now, $\dfrac{h}{BD}=\tan 45°=1 \quad \therefore \quad BD=h$

and $\dfrac{h}{BC}=\tan 60°=\sqrt{3} \quad \therefore \quad BC=h/\sqrt{3}$

$\therefore \quad BD-BC=h-\dfrac{h}{\sqrt{3}}$

or, $40=h\left(1-\dfrac{1}{\sqrt{3}}\right)$

$=h\dfrac{\sqrt{3}-1}{\sqrt{3}}$

$\therefore \quad h=\dfrac{40\sqrt{3}}{\sqrt{3}-1}=\dfrac{40\times 1{\cdot}732}{0{\cdot}732}$ ft.

$=94{\cdot}64$ ft.

Ex. 5. *From the foot of a tower the angle of elevation of the top of a column is 60° and from the top of the tower, which is 50 ft. high, the angle of elevation is 30°. Find the height of the column.*

Let AB be the column and CD the tower. Draw CE parallel to BD. Let AB $=x$ ft. and BD $=y$ ft.

Now, $\frac{AB}{BD}=\tan 60°$.

or, $\frac{x}{y}=\sqrt{3}$ $\therefore$ $x=\sqrt{3}y$......(1)

Again, $\frac{AE}{EC}=\tan 30°$

or, $\frac{x-50}{y}=\frac{1}{\sqrt{3}}$ $\therefore$ $y=\sqrt{3}(x-50)$ ...(2)

$\therefore$ from (1) & (2), $x=3(x-50)$

or, $x=75$.

So the height of the column is 75 ft.

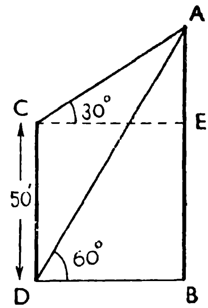


Fig. 52

Ex. 6. *The angles of depression and elevation of the top of a tower 100 ft high from the top and bottom of another tower are 60° and 25° respectively; calculate the height of the second tower (Given cot 25° = 2·144).*

AB is the tower of 100 ft. height; PQ is the second tower. PX is a horizontal line parallel to BQ.

$\therefore$ $\angle APX=60°$

and $\angle AQB=25°$

Now, $QB=AB \cot 25°$

$PC=CA \tan 60°$

$=QB \tan 60°$

$=AB \cot 25° \tan 60°$

$\therefore$ $PQ=PC+CQ$

$=PC+AB$

$=AB(\cot 25° \tan 60°+1)$

$=100(2{\cdot}144\times\sqrt{3}+1)$

$=214{\cdot}4.\sqrt{3}+100$

$=214{\cdot}4\times 1{\cdot}732+100$

$=371{\cdot}34+100$

$=471{\cdot}34$ ft.

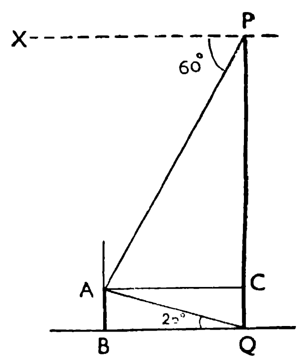


Fig. 53

Ex. 7. *Two tower stands on a horizontal plane and their distance apart is 120 ft. A person standing successively at the bases observes that the angular elevation of one is double that of the other, but when half way between them, their elevations appear to be complementary. Show that the heights are 90 fit. and 40 ft. respectively.* [P. U. 1939 (Supp.)]

Here AB and CD are the two towers of height y and x ft. respectively. The elevation of the top of CD from the base

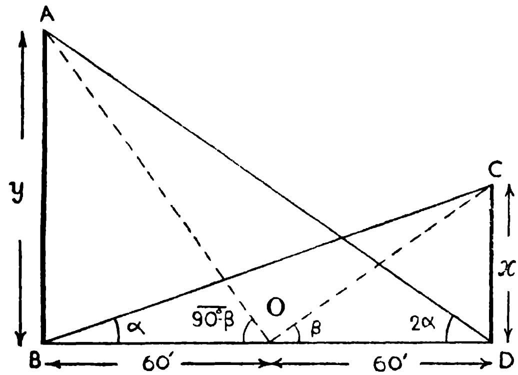


Fig. 54

B of AB is α ; that of the top of AB from the base D of CD is 2α.

From the point O, just half way between the towers, the elevations of C and A are respectively β and $90°-\beta$.

Now, $CD=BD \tan \alpha$, $\therefore$ $x=120 \tan \alpha$
$AB=BD \tan 2\alpha$, $\therefore$ $y=120 \tan 2\alpha$.

$$\therefore \quad y=\frac{120.\ 2 \tan \alpha}{1-\tan^2 \alpha}=\frac{2x}{1-(x/120)^2} \quad \text{(i)} (\because \tan\alpha=x/120)$$

Again, $CD=OD \tan \beta$ $\therefore$ $x=60 \tan \beta$
and, $AB=BO \tan (90°-\beta)$ $\therefore$ $y=60 \cot \beta$.

$$\therefore \quad y=60/\tan \beta=60/\frac{x}{60}=3600/x. \quad \text{...(ii)}$$

$(\because \tan \beta=x/60)$

From (i) and (ii), $\dfrac{3600}{x}=\dfrac{2x}{1-x^2/14400}$

or, $\dfrac{3600}{x}=\dfrac{2x.\ 14400}{14400-x^2}$

or, $\dfrac{1}{x}=\dfrac{8x}{14400-x^2}$

or, $9x^2=14400$

$\therefore\quad x=40$ ft.

$\therefore\quad y=90$ ft.

Ex. 8. *The angular elevation of a tower at a place due south of it is* α, *and at another place due west of the first and distant* d *from it the elevation is* β ; *prove that the height of the tower is* $d/\sqrt{\cot^2\beta-\cot^2\alpha}$.

Let AB be the tower of height x ; let C be the first point of observation due south of AB.

$\therefore$ $\triangle$ABC is a rt-angled triangle. Suppose BC$=y$. D is the second point of observation due west of C and distant d from it. So, $\triangle$ ABD is also rt. angled. Join DC. Then $\triangle$ DCB is also rt. angled, DB being the hypotenuse. $\angle$ADB$=\beta$, $\angle$ACB$=\alpha$.

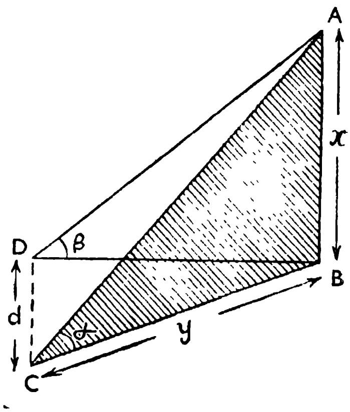


Fig. 55

Now, from geometry,

DB^2 $=$DC^2 $+$BC^2

$=d^2+y^2$ ....(1)

In $\triangle$ABC, $\cot\alpha=y/x$ ....(2).

In $\triangle$ADB, $\cot\beta=$DB$/x$ ....(3)

Squaring (2) and (3) and using (1),

$$\cot^2\beta-\cot^2\alpha=\frac{d^2+y^2}{x^2}-\frac{y^2}{x^2}=\frac{d^2}{x^2}$$

$$\therefore\quad x^2=\frac{d^2}{\cot^2\beta-\cot^2\alpha}$$

$$\therefore\quad x=\frac{d}{\sqrt{\cot^2\beta-\cot^2\alpha}}.$$

Examples 14

1. An observer at a distance of 40 ft. from the foot of a a palm tree observes the angle of elevation of the top of the tree to be 60°. Calculate its height.

2. A circular tank has a pole standing vertically out of its centre, whose top is 150 feet above the water level. At a point in the circumference the angle subtended by the pole is 60°. What is the radius of the pond ?

3. The shadow of a tower 200 feet long is $200\sqrt{3}$ ft. in length. What is the elevation of the sun ?

4. A man observes the elevation of a tower to be 30° ; advancing towards the tower a distance of 300 ft., he notices the new elevation to be 60° ; find the height of the tower.

5. A ladder 45 ft. long just reaches the top of a wall. If the ladder makes an angle 60° with the wall, find the height of the wall and the distance of the foot of the ladder from the wall.

6. One chimney is 30 ft. higher than another. A man standing at a distance of 100 ft. from the lower observes their tops to be in a line inclined at an angle of 27°2′ to the horizon. Find their heights, given tan 27°2′ = ·51.

7. The elevation of a hill from a place due east of it is 45° and at a second place due south of the previous, the elevation is 30°. If the distance between the two places of observations be 400 yds., find the height of the hill.

8. The elevation of a steeple at a place due south of it is 45° and at another place due west of the former is 30°. If the distance between the two places be a, find the height of the steeple. [P. U. (supp). 1944]

9. The shadow of a tower standing on a level plane is 100 ft. longer when the altitude is 30° than when it is 45°. Find the height of the tower.

10. A man walking along a straight road observes at a milestone a house in a direction of 30° with the road. At the

next he sees it in a direction of 60°. Find the distance of the house from the road.

11. A man at a certain point on the bank of the Ganges observes that the elevation of a tree on the opposite bank and directly opposite him is 60°, his eye being level with the ground. He then retires 100 ft. and sees that the elevation is 45°. Find the breadth of the Ganges there and the height of the tree.

12. From the top of a cliff 150 ft. high, angles of depression of two boats which are due south of the observer are 15° and 75°. Find their distance apart, having given $\cot 15° = 2 + \sqrt{3}$, $\cot 75° = 2 - \sqrt{3}$.

13. The angles of depression of the top and foot of a tower seen from a monument 196 ft. high are 30° and 60° respectively. What is the height of the tower ?

14. A pole stands upon the top of a mansion. At a distance of 60 ft. the angles of elevation of the tops of the pole and the mansion are found to be 60° and 30° respectively. Find the length of the pole.

15. At the foot of a mountain the elevation of its summit is found to be 45° ; after ascending 2 miles towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain.

16. The elevation of a tower due north of a station A is θ and at a station B due west of A is ϕ. Show that the altitude of the tower is

$$\frac{AB \sin \theta \sin \phi}{\sqrt{\sin^2 \theta - \sin^2 \phi}}$$

17. A valley is crossed by a horizontal bridge of length l. The sides of the valley make angles α and β with the horizon ; prove that the height of the bridge above the bottom of the valley is

$$\frac{l}{\cot \alpha + \cot \beta}$$

18. The height of a lighted candle is 6 inches and the radius of its section $\frac{3}{4}$ of an inch. The radius of the shadow cast by it is $5\frac{1}{4}$ inches. Find the height of the flame, and if the inclination to the horizon of the line joining the top of the flame to a point on the boundary of the shadow is α, show that $\tan \alpha = 4/3$.

APPENDICES

I. Elimination

1-1. When an unknown quantity appears in each of two simultaneous equations, the values of the unknown quantity obtained by solving the first equation must satisfy the second. When these values are substituted we get a relation free from the unknown quantities. This relation is called the *eliminant* of the two equations and the process of obtaining it is called the *elimination.* In general, we can eliminate any number of unknowns, if the number of equations connecting them be one more than their number.

No general rule can, however, be laid down for the elimination of quantity or quantities from two or more trigonometrical equations. The form of the equation will generally suggest the method to be employed ; the usual algebraic artifices and the identical relations existing between the trigonometrical functions would help to a great extent.

1-2. Illustrated examples

Ex. 1. *Eliminate θ from $x=a\cos\theta$ and $y=b\sin\theta$*

The first equation gives $\cos\theta=x/a$

and the second gives $\sin\theta=y/b$

But, since, $\cos^2\theta+\sin^2\theta=1$,

$$\therefore\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

which is the required eliminant.

Ex. 2. *To eliminate θ between the equations*

$$\cos\theta+\sin\theta=a$$
$$\cos\theta-\sin\theta=b$$

Squaring and adding the above equations,

$$a^2+b^2=(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2$$
$$=2(\cos^2\theta+\sin^2\theta)$$

or, $a^2+b^2=2$, which is the required eliminant.

Ex. 3. *Eliminate θ from the equations*

$$a \sec\theta + b\tan\theta + c = 0$$
$$a'\sec\theta + b'\tan\theta + c' = 0$$

By cross-multiplication,

$$\frac{\sec\theta}{bc'-b'c} = \frac{\tan\theta}{ca'-c'a} = \frac{1}{ab'-a'b}.$$

$\therefore$ $\sec\theta = (bc'-b'c)/(ab'-a'b)$

$\tan\theta = (ca'-c'a)/(ab'-a'b)$

But, $\sec^2\theta - \tan^2\theta = 1.$

$\therefore$ $(bc'-b'c)^2 - (ca'-c'a)^2 = (ab'-a'b)^2$

which is the required eliminant.

Ex. 4. *Eliminate θ and ϕ between the equations*

$$a\sin^2\theta + b\cos^2\theta = c \qquad \ldots\ldots\text{(i)}$$
$$b\sin^2\phi + a\cos^2\phi = d \qquad \ldots\ldots\text{(ii)}$$
$$a\tan\theta = b\tan\phi \qquad \ldots\ldots\text{(iii)}$$

From (i), $a\sin^2\theta + b\cos^2\theta = c(\sin^2\theta + \cos^2\theta)$

or, $(a-c)\sin^2\theta = (c-b)\cos^2\theta$

or, $\tan^2\theta = (c-b)/(a-c)$

From (ii), similarly, $\tan^2\phi = (d-a)/(b-d)$

From (iii), $a^2\tan^2\theta = b^2\tan^2\phi$

or, $a^2(c-b)/(a-c) = b^2(d-a)/(b-d)$

or, $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{1}{d}$ after simplification.

This, therefore, is the required eliminant.

Ex. 5. *Eliminante θ from the equations*

$$x\sin\theta + y\cos\theta = 2a\sin 2\theta$$
$$x\cos\theta - y\sin\theta = a\cos 2\theta$$

Solving for x and y, we have,

$$x = a\cos\theta\,(2-\cos 2\theta)$$
$$y = a\sin\theta\,(2+\cos 2\theta)$$

or, $x=a\cos\theta\,(\cos^2\theta+3\sin^2\theta)$
$y=a\sin\theta\,(3\cos^2\theta+\sin^2\theta)$

$\therefore\quad x+y=a\,(\cos\theta+\sin\theta)^3$
$x-y=a\,(\cos\theta-\sin\theta)^3$

$\therefore\quad (x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$, which is the required eliminant.

Ex. 6. *Eliminate θ and ϕ from the equations*

$$x\cos\theta+y\sin\theta=2a$$
$$x\cos\phi+y\sin\phi=2a$$
$$2\sin\theta/2\sin\phi/2=1$$

From the given conditions, θ and ϕ are the roots of the equation,

$$x\cos\alpha+y\sin\alpha=2a\ ;$$

or, $(x\cos\alpha-2a)^2=y^2\sin^2\alpha=y^2(1-\cos^2\alpha)$;

$\therefore\quad (x^2+y^2)\cos^2\alpha-4ax\cos\alpha+4a^2-y^2=0.$

This is a quadratic in $\cos\alpha$ with roots $\cos\theta$ and $\cos\phi$.
Again, $1=4\sin^2\theta/2\sin^2\phi/2=(1-\cos\theta)(1-\cos\phi)$;
whence, $\cos\theta+\cos\phi=\cos\theta\cos\phi$

$$\therefore\quad \frac{4ax}{x^2+y^2}=\frac{4a^2-y^2}{x^2+y^2}$$

$\therefore\quad y^2=4a(a-x)$, which is the required eliminant.

Examples A 1

Eliminate θ from the following pair of equations

1. $x=a\sec\theta,\ y=b\tan\theta$
2. $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,\quad \frac{x}{a}\sin\theta-\frac{y}{b}\cos\theta=1$
3. $\cos\theta+\sin\theta=a,\ \cos 2\theta=b$
4. $x=\cot\theta+\tan\theta,\ y=\operatorname{cosec}\theta-\sin\theta$
5. $l\cos\theta+m\sin\theta+n=0$
   $p\cos\theta+q\sin\theta+r=0$

6. $\cos\theta+\sin\theta\sin 2\theta=a$
$\sin\theta+\cos\theta\sin 2\theta=b$

7. $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=c^2$ and $l\tan\theta=m$

Eliminate θ and ϕ from the following equations

8. $a\sin^2\theta+b\cos^2\theta=1$
$a\cos^2\phi+b\sin^2\phi=1$
$a\tan\theta=b\tan\phi$

9. $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=1$
$\dfrac{x}{a}\cos\phi+\dfrac{y}{b}\sin\phi=1$
$\theta-\phi=c$

10. $\tan\theta+\tan\phi=p$
$\cot\theta+\cot\phi=q$
$\theta-\phi=r$

11. $a\cos^2\theta+b\sin^2\theta=m\cos^2\phi$
$a\sin^2\theta+b\cos^2\theta=n\sin^2\phi$
$m\tan^2\theta=n\tan^2\phi$

12. If $\sin\phi+\cos\phi=l$ and $\sin 2\phi+\cos 2\phi=m$
show that $(l^2-m-1)^2=l^2(2-l^2)$

13. If $\cos\theta-\sin\theta=x$ and $\cos 3\theta+\sin 3\theta=y$
show that $x=3y-2y^3$

14. If $\dfrac{\tan(\alpha+\beta)}{\tan(\alpha-\beta)}=\dfrac{a+b}{a-b}$ and $a\cos 2\beta+b\cos 2\alpha=c$
show that $a^2+c^2-2ac\cos 2\beta=b^2$

II Summation of finite series

2-1. Any expression in which the successive terms are found to obey some regular law is a series. If the series terminates at some term it is said to be **finite**, if not **infinite**.

2-2. Sum of sines of n angles in A. P.

Let the series be denoted by

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\dots\dots\dots+\sin\{\alpha+\overline{n-1}\beta\}$$

and let S be its sum, α the first angle, β the common difference of the angles.

Multiplying each term of the series by 2 sin (half. diff.) i.e., by 2 sin $\beta/2$, we get,

$$2\sin\alpha.\sin\frac{\beta}{2}=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right)$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2}=\cos\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{3\beta}{2}\right)$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2}=\cos\left(\alpha+\frac{3\beta}{2}\right)-\cos\left(\alpha+\frac{5\beta}{2}\right)$$

$$\dots\dots\dots=\dots\dots\dots$$

$$2\sin\{\alpha+\overline{n-1}\beta\}\sin\frac{\beta}{2}=\cos\left(\alpha+\frac{2n-3}{2}\beta\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right).$$

By addition,

$$2S\sin\frac{\beta}{2}=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)$$

$$=2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2};$$

$$\therefore\quad S=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha+\frac{n-1}{2}\beta\right)\qquad\dots\dots\dots(\text{I})$$

Cor. In a similar manner we may show that the sum of the cosine series

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\dots\dots+\cos\{\alpha+\overline{n-1}\beta\}$$

$$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha+\frac{n-1}{2}\beta\right)\qquad\dots\dots\dots(\text{II})$$

Note : If $\sin\frac{n\beta}{2}=0$, the expressions (I) & (II) both vanish. In that case $\frac{n\beta}{2}=k\pi$ or, $\beta=\frac{2k\pi}{n}$, where k is an integer.

Hence the sum of the sines and cosines of n angles in arithmetical progression are each equal to zero when the common difference of the angles is an even multiple of π/n.

2-3. The above formulæ may be expressed for convenience as follows.

Sum of sines of n angles in A. P.

$$=\frac{\sin\frac{\text{n.diff.}}{2}}{\sin\frac{\text{diff.}}{2}}\sin\frac{\text{first angle}+\text{last angle}}{2}$$

Sum of cosines of n angles in A. P.

$$=\frac{\sin\frac{\text{n. diff.}}{2}}{\sin\frac{\text{diff.}}{2}}\cos\frac{\text{first angle}+\text{last angle}}{2}$$

2-4. Illustrated examples

Ex. 1. *Find the sum of the series*

$$\sin\alpha+\sin 3\alpha+\sin 5\alpha+\cdots$$

Here the common difference of the angles is 2α,

$$\therefore \quad \text{Sum}=\frac{\sin n\alpha}{\sin\alpha}\sin\frac{\alpha+(2n-1)\alpha}{2}$$

$$=\frac{\sin n\alpha \sin n\alpha}{\sin\alpha}$$

$$=\frac{\sin^2 n\alpha}{\sin\alpha}$$

Ex. 2. *Find the sum of the series*

$$\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)-\cos(\alpha+3\beta)+\ldots$$

[Some series may be brought by simple transformation under the above forms. Here is an example.]

The above series is equal to

$$\cos\alpha+\cos(\pi+\alpha+\beta)+\cos(2\pi+\alpha+2\beta)+\cos(3\pi+\alpha+3\beta)+\ldots$$

Here, now, the common difference of the angles is $\pi+\beta$ and the last angle is $\alpha+(n-1)(\beta+\pi)$

$$\therefore \quad \text{Sum}=\frac{\sin\frac{n(\beta+\pi)}{2}}{\sin\frac{\beta+\pi}{2}}\cos\left\{\alpha+\frac{(n-1)(\beta+\pi)}{2}\right\}$$

Ex. 3. *Find the sum of the series*

$$cosec\ \alpha+cosec\ 2\alpha+cosec\ 4\alpha+\ldots+cosec\ 2^{n-1}\alpha.$$

[This example is an illustration of the method of difference ; if the rth term of a series can be expressed as the difference of two quantities one of which is the same function of r as the other is of $(r+1)$, the series may be summed quite readily.]

$$\text{Cosec}\ \alpha=\frac{1}{\sin\alpha}=\frac{\sin\alpha/2}{\sin\alpha/2\ \sin\alpha}=\frac{\sin\ (\alpha-\alpha/2)}{\sin\alpha/2\ \sin\alpha}=\cot\frac{\alpha}{2}-\cot\alpha$$

$$\therefore \quad \text{cosec}\ \alpha=\cot\alpha/2-\cot\alpha$$

Similarly, $\text{cosec}\ 2\alpha=\cot\alpha-\cot 2\alpha$

$$\text{cosec}\ 4\alpha=\cot 2\alpha-\cot 4\alpha$$

$$\ldots\ \ldots = \ldots\ \ldots$$

$$\text{cosec}\ 2^{n-1}\alpha=\cot 2^{n-2}\alpha-\cot 2^{n-1}\alpha$$

By adding, $\quad S=\cot\frac{\alpha}{2}-\cot 2^{n-1}\alpha$

Ex. 4. *Find the sum of the series*

$$tan^{-1}\frac{x}{1+1.\ 2.\ x^2}+tan^{-1}\frac{x}{1+2.\ 3.\ x^2}+\cdots$$

$$+tan^{-1}\frac{x}{1+n(n+1)x^2}.$$

Since. $\tan^{-1}\dfrac{x}{1+r(r+1)x^2}=\tan^{-1}(r+1)x-\tan^{-1}rx$

$\therefore$ 1st term $=\tan^{-1} 2x-\tan^{-1}x$

2nd term $=\tan^{-1} 3x-\tan^{-1} 2x$

3rd term $=\tan^{-1} 4x-\tan^{-1} 3x$

$\ldots\ \ldots = \ldots\ \ldots$

nth term $=\tan^{-1}(n+1)x-\tan^{-1} nx$

$\therefore$ $S=\tan^{-1}(n+1)x-\tan^{-1}x$

Ex. 5. *Find the sum of the series*

$$\sin^2\theta+\sin^2 2\theta+\sin^2 3\theta+\ldots+\sin^2 n\theta$$

[The example is important in the sense that here we shall find the sum of the squares of the sines of a series of angles in A. P. by using identities. Similarly, the cubes of sines as well as the squares and cubes of cosines of a series of angles in A. P. may be summed.]

Since, $2\sin^2\theta=(1-\cos 2\theta)$

$2\sin^2 2\theta=(1-\cos 4\theta)$ etc.

$$\therefore\quad 2S=(1-\cos 2\theta)+(1-\cos 4\theta)+\ldots+(1-\cos 2n\theta)$$

$$=n-(\cos 2\theta+\cos 4\theta+\ldots+\cos 2n\theta)$$

$$=n-\frac{\sin n\theta}{\sin\theta}\cos(n+1)\theta$$

where S is the sum of the series.

$$\therefore\quad S=\frac{n}{2}-\frac{1}{2}\cdot\frac{\sin n\theta}{\sin\theta}\cos(n+1)\theta$$

Examples A 2

Sum the following series to n terms

1. $\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots\ldots$
2. $\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots\ldots$
3. $\sin\alpha-\sin 2\alpha+\sin 3\alpha-\ldots\ldots$
4. $\cos\alpha-\cos 2\alpha+\cos 3\alpha-\ldots\ldots$
5. $\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\ldots\ldots$
6. $\cos^2\theta+\cos^2 2\theta+\cos^2 3\theta+\ldots\ldots$
7. $\sin^3\alpha+\sin^3 3\alpha+\sin^3 5\alpha+\ldots\ldots$
8. $\cos^3\alpha+\cos^3 3\alpha+\cos^3 5\alpha+\ldots\ldots$
9. $\sin^2\theta-\sin^2 2\theta+\sin^2 3\theta-\sin^2 4\theta+\ldots\ldots$
10. $\cos^2\theta-\cos^2 2\theta+\cos^2 3\theta-\cos^2 4\theta+\ldots\ldots$
11. $\cos\theta-\sin 2\theta-\cos 3\theta+\sin 4\theta+\cos 5\theta-\sin 6\theta-\ldots\ldots$
12. $\cos\alpha+2\cos(\alpha+\beta)+3\cos(\alpha+2\beta)+\ldots\ldots$
13. $\sin\alpha+2\sin(\alpha+\beta)+3\sin(\alpha+2\beta)+\ldots\ldots$
14. $\sin\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\sin 3\alpha\sin 4\alpha+\ldots\ldots$

15. $\tan\alpha+2\tan 2\alpha+4\tan 4\alpha+8\tan 8\alpha+\ldots\ldots$

16. $\dfrac{\sin x}{\cos 3x}+\dfrac{\sin 3x}{\cos 3^2x}+\dfrac{\sin 3^2x}{\cos 3^3x}+\ldots\ldots$

17. $\tan^{-1}\dfrac{1}{1+1+1^2}+\tan^{-1}\dfrac{1}{1+2+2^2}+\tan^{-1}\dfrac{1}{1+3+3^2}+\ldots$

18. *Show that*

(i) $\dfrac{\sin\ \ +\sin 3\theta+\sin 5\theta+\ldots\ldots\text{to } n \text{ terms}}{\cos\theta+\cos 3\theta+\cos 5\theta+\ldots\ldots\text{to } n \text{ terms}}=\tan n\theta$

(ii) $\sin^2\alpha+\sin^2\left(\alpha+2\frac{\pi}{n}\right)+\sin^2\left(\alpha+4\frac{\pi}{n}\right)$ to n terms $=\frac{1}{2}n$

III. Miscellaneous theorems

3-1. To find the expansion of tan (A—B) geometrically

Let the $\angle XOY=A$ and $\angle YOZ=B$, where A and B are both positive. $\therefore\ \angle XOZ=A+B$

Take any point P on the line OZ and drop PM and PN perps. to OX and OY in order. Drop also perps. NR and NS

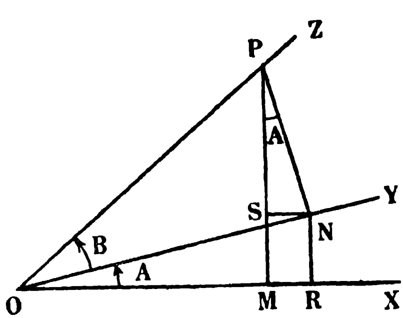


Fig. 54

to OX and PM respectively. Now, $\angle SPN$=complement of $\angle PNS=\angle SNO=A$.

$$\therefore\quad \tan(A+B)=\frac{PM}{OM}=\frac{NR+PS}{OR-SN}.$$

$$=\frac{NR/OR+PS/OR}{1-SN/OR}$$

$$=\frac{NR/OR+PS/OR}{1-\dfrac{SN.\ PS}{PS.\ OR}}$$

Now, $NR/OR=\tan A$, and $SN/SP=\tan A$;
and, from similar triangles NOR and SPN,

$$\frac{PS}{OR}=\frac{PN}{ON}=\tan B.$$

$$\therefore\quad \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan A}.$$

Note : In a similar manner, with the help of the figure 26, Art 5-2, we can find the expansion of **tan (A—B)**. This is left as an exercise to the students.

3-2. To prove geometrically the formulæ for the sine, cosine and tangent of (A+B) for angles of any magnitude

Let the line OX revolving counter-clockwise reach the posi-

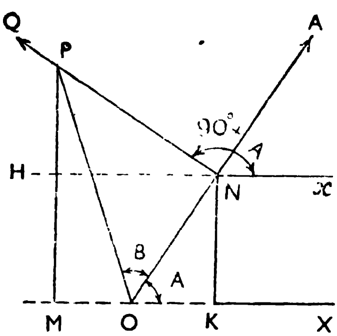


Fig. 55

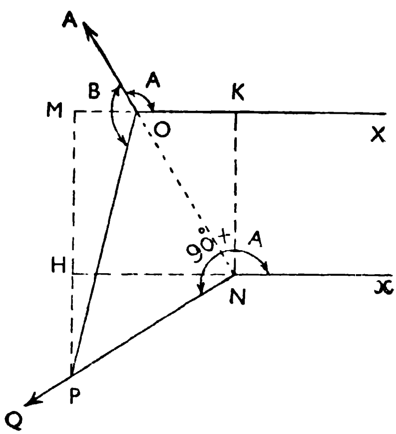


Fig. 56

tion OA and described the angle A. Thereafter, it revolves in the same direction and generates the angle $B=\angle AOP$.

$$\therefore\quad \angle XOP=A+B$$

Take any point P on the line OP bounding the compound angle A+B and dorp perps. PM and PN to OX and OA (produced, if needed) respectively. Also drop perp. NK to OX. Draw Nx || OX and in the same sense and let it meet MP (produced, if required) in H. Let the $\angle$ANQ is a rt. angle described in the positive direction from NA.

$$\therefore \quad \angle x\text{NQ (described positively from N}x) \\ = x\text{NA}+90^\circ \\ = \text{A}+90^\circ$$

So, $\frac{\text{NH}}{\text{NP}}=\cos(\text{A}+90^\circ)$; $\frac{\text{HP}}{\text{NP}}=\sin(\text{A}+90^\circ)$

the positive directions of NH, HP are in conformity with our convention.

Also, $\frac{\text{ON}}{\text{OP}}=\cos \text{B}$, $\frac{\text{NP}}{\text{OP}}=\sin \text{B}$, $\frac{\text{NP}}{\text{ON}}=\tan \text{B}$

ON, NP are positive with respect to $\angle$B, when they lie along OA and NQ in order, negative when along AO and QN respectively.

Hence, maintaining the directions by the order of the letters,

$$\sin(\text{A}+\text{B})=\frac{\text{MP}}{\text{OP}}=\frac{\text{MH}+\text{HP}}{\text{OP}}=\frac{\text{KN}}{\text{OP}}+\frac{\text{HP}}{\text{OP}}$$

$$=\frac{\text{KN}}{\text{ON}}\cdot\frac{\text{ON}}{\text{OP}}+\frac{\text{HN}}{\text{NP}}\cdot\frac{\text{NP}}{\text{OP}}$$

$$=\sin \text{A} \cos \text{B}+\sin(90^\circ+\text{A}) \sin \text{B}$$

$$=\sin \text{A} \cos \text{B}+\cos \text{A} \sin \text{B}$$

$$\cos(\text{A}+\text{B})=\frac{\text{OM}}{\text{OP}}=\frac{\text{OK}+\text{KM}}{\text{OP}}=\frac{\text{OK}}{\text{OP}}+\frac{\text{NH}}{\text{OP}}$$

$$=\frac{\text{OK}}{\text{ON}}\cdot\frac{\text{ON}}{\text{OP}}+\frac{\text{NH}}{\text{NP}}\cdot\frac{\text{NP}}{\text{OP}}$$

$$=\cos \text{A} \cos \text{B}+\cos(90^\circ+\text{A}) \sin \text{B}$$

$$=\cos \text{A} \cos \text{B}-\sin \text{A} \sin \text{B}$$

$$\tan(A+B)=\frac{MP}{OM}=\frac{MH+HP}{OK+KM}=\frac{KN+HP}{OK+NH}$$

$$=\frac{\dfrac{KN}{OM}+\dfrac{HP}{OK}}{1+\dfrac{NH}{HP}\cdot\dfrac{HP}{OK}}$$

Now, $\dfrac{HP}{NP}=\sin(90°+A)=\cos A=\dfrac{OK}{ON}$, algebraically.

$$\therefore\quad \frac{HP}{OK}=\frac{NP}{ON}=\tan B,\ \frac{KN}{OK}=\tan A,$$

and $\dfrac{NH}{HP}=\cot(90°+A)=-\tan A.$

$$\therefore\quad \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

[In the second fig. B is obtuse ; N lies on AO produced. So ON, here, is a negative length. Thus algebraically, still $\sin A=KN/ON$, $\cos A=OK/ON$, $\tan A=KN/OK$.]

3-3. To prove the formulæ for the sine, cosine and tangent of (A – B) for angles of any magnitude

Let $\angle XOA=A$, described counter-clockwise.

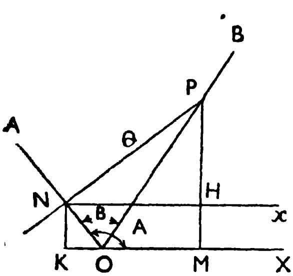


Fig. 57

$\angle AOB=B$, described clockwise.

$\therefore\quad \angle XOB=A-B.$

From a point P on the bounding line OB, drop perps. as shown. Draw $Nx \parallel OX$ and in the sense of OX and let $\angle ANQ$ is a rt. angle described clockwise from NA.

$\therefore\quad \angle xNQ=A-90°.$

With usual convention regarding signs,

$$\sin(A-B)=\frac{MP}{OP}=\frac{KN+HP}{OP}=\frac{KN}{ON}\cdot\frac{ON}{OP}+\frac{HP}{NP}\cdot\frac{NP}{OP}$$

$$=\sin A\cos B+\sin(A-90°).\sin B$$

$$=\mathbf{\sin A\cos B-\cos A\sin B}$$

$$\cos(A-B)=\frac{OM}{OP}=\frac{OK+KM}{OP} \quad \text{(algebraically)}$$

$$=\frac{OK}{ON}\cdot\frac{ON}{OP}+\frac{NH}{NP}\cdot\frac{NP}{OP}$$

$$=\cos A\cos B+\cos(A-90°)\sin B$$

$$=\cos A\cos B+\sin A\sin B$$

$$\tan(A-B)=\frac{MP}{OM}=\frac{KN+HP}{OK+KM}=\frac{\frac{KN}{OK}+\frac{HP}{OK}}{1+\frac{NH}{HP}\cdot\frac{HP}{OK}}$$

Also, $\frac{HP}{NP}=\sin(A-90°)=-\cos A=-\frac{OK}{ON}$, algebraically.

$\therefore \quad \frac{HP}{OK}=-\frac{NP}{ON}=-\tan B$; $\frac{NH}{HP}=\cot(A-90°)=-\tan A$

$$\therefore \quad \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

3-4. Geometrical proof of the 2A formulæ

Let BCA be the diameter of the semicircle BPA with centre C. On the circumference, take any point P and join PB, PC, PA. Drop PM perp. to BA.

Let $\angle PBA=A$ $\therefore$ $\angle ACP=2A$.

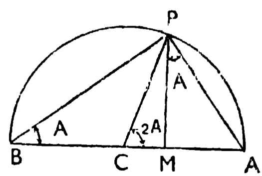


Fig. 58

Now, $\sin 2A=\frac{MP}{CP}=\frac{MP}{CA}=2\frac{MP}{BA}$

$$=2\frac{MP}{BP}\cdot\frac{BP}{BA}$$

$$=2\sin A\cos A.$$

$$\cos 2A=\frac{CM}{CP}=2\frac{CM}{BA}=\frac{CM+CM}{BA}$$

$$=\frac{(BM-BC)+(CA-MA)}{BA}$$

$$=\frac{BM-MA}{BA}=\frac{BM}{BA}-\frac{MA}{BA}$$

$$=\frac{BM}{BP}\cdot\frac{BP}{BA}-\frac{MA}{PA}\cdot\frac{PA}{BA}$$

$$=\cos^2 A-\sin^2 A.$$

The cos 2A formula has two other forms which may be derived as follows.

$$\cos 2A = \frac{CM}{CP} = \frac{BM - BC}{BC} = 2\,\frac{BM}{BA} - 1$$

$$= 2\,\frac{BM}{BP} \cdot \frac{BP}{BA} - 1$$

$$= 2\cos^2 A - 1$$

Again, $$\cos 2A = \frac{CM}{CP} = \frac{CA - MA}{CA} = 1 - \frac{MA}{CA}$$

$$= 1 - 2.\,\frac{MA}{BA} = 1 - 2\,\frac{MA}{PA} \cdot \frac{PA}{BA}$$

$$= 1 - 2\sin^2 A$$

$$\tan 2A = \frac{PM}{CM} = \frac{2\,PM}{2\,CM} = \frac{2\,PM}{BM - AM}$$

$$= \frac{2\,PM/BM}{1 - \dfrac{AM}{BM}} = \frac{2\,PM/BM}{1 - \dfrac{AM}{PM} \cdot \dfrac{PM}{BM}}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

3-5. To prove the sum and difference formulæ by geometry

Let $\angle EOF = A$, $\angle EOG = B$. Let OL bisect $\angle KOH$ and

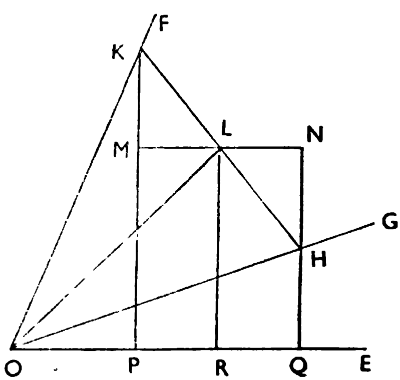


Fig. 59

through L draw KLH perp. to OL. Draw KP, HQ, LR perps.

to OE and through L draw MLN parallel to OE meeting KP in M and QH in N.

Now, $\triangle$ MKL $\equiv \triangle$ NHL,

$\therefore$ KM=NH, ML=LN, PR=RQ.

Also, $\angle KOL = \angle LOH = \frac{1}{2}(A-B)$

$\angle LOR = B + \frac{1}{2}(A-B) = \frac{1}{2}(A+B)$

$$\therefore \quad \sin A + \sin B = \frac{KP}{OK} + \frac{HQ}{OH} = \frac{KP+HQ}{OK} \quad (\because \quad OK = OH)$$

$$= \frac{(KM+LR)+(LR-NH)}{OK}$$

$$= 2\frac{LR}{OK} = 2\frac{LR}{OL} \cdot \frac{OL}{OK}$$

$$= 2 \sin ROL \cos KOL$$

$$= 2 \sin\frac{A+B}{2} \cos\frac{A-B}{2}.$$

$$\cos A + \cos B = \frac{OP}{OK} + \frac{OQ}{OH} = \frac{OP+OQ}{OK}$$

$$= \frac{(OR-PR)+(OR+RQ)}{OK}$$

$$= 2\frac{OR}{OK} = 2\frac{OR}{OL} \cdot \frac{OL}{OK}$$

$$= 2 \cos ROL \cos KOL$$

$$= 2 \cos\frac{A+B}{2} \cos\frac{A-B}{2}$$

Note: The derivation of the expressions for **sin A—sin B** and **cos A—cos B** is left as an exercise to the students.

3-6. Two important trigonometrical relations

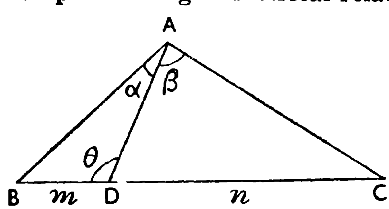


Fig. 60

Let D be a point on the base BC of the $\triangle$ABC such that

AD divides BC into two parts m and n and the angle A into two parts α, β. Let also the $\angle ADB=\theta$.

To prove

(i) $(m+n)\cot\theta=n\cot\beta-m\cot\alpha$

(ii) $(m+n)\cot\theta=m\cot C-n\cot B$

$$\frac{m}{n}=\frac{BD}{DC}=\frac{BD}{AD}\cdot\frac{AD}{DC}=\frac{\sin BAD}{\sin ABD}\cdot\frac{\sin ACD}{\sin DAC}\quad\ldots\ldots(1)$$

$$=\frac{\sin\alpha}{\sin(\theta+\alpha)}\cdot\frac{\sin(\theta-\beta)}{\sin\beta}\quad\left[\because\ \angle ABD=\pi-(\alpha+\theta),\ \angle ACD=\theta-\beta\right]$$

$$=\frac{\sin\alpha(\sin\theta\cos\beta-\cos\theta\sin\beta)}{\sin\beta(\sin\theta\cos\alpha+\cos\theta\sin\alpha)}$$

$$=\frac{\cot\beta-\cot\theta}{\cot\alpha+\cot\theta}\quad\text{(Dividing num. and denom. by }\sin\alpha\sin\beta\sin\theta)$$

$\therefore\ (m+n)\cot\theta=n\cot\beta-m\cot\alpha$

Again, starting from (1),

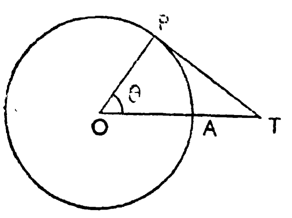
$$\frac{m}{n}=\frac{\sin(\theta+B)}{\sin B}\cdot\frac{\sin C}{\sin(\theta-C)}\quad\left[\because\ \angle BAD=\pi-(\theta+B),\ \angle DAC=\theta-C\right]$$

$$=\frac{\sin C(\sin\theta\cos B+\cos\theta\sin B)}{\sin B(\sin\theta\cos C-\cos\theta\sin C)}$$

$$=\frac{\cot B+\cot\theta}{\cot C-\cot\theta}\quad\text{(Dividing num. and denom. by }\sin B\sin C\sin\theta\ )$$

$\therefore\ (m+n)\cot\theta=m\cot C-n\cot B$

3-7. To prove $\sin\theta<\theta<\tan\theta$, where θ is the circular measure of any positive acute angle.



Fig, 61

Let the angle AOP be θ radians. With O as centre and OA as radius draw a circle. Draw PT the tangent at P to meet OA produced at T.

Let r be the radius of the circle.

Now, the area of $\triangle AOP=\frac{1}{2}$ AO. OP sin AOP$=\frac{1}{2}r^2\sin\theta$
the area of sector AOP$=\frac{1}{2}r^2\theta$.
the area of $\triangle OPT=\frac{1}{2}$ OP. PT$=\frac{1}{2}r.\tan\theta=\frac{1}{2}r^2\tan\theta$.

But, obviously, from the figure,

$$\triangle APO < \text{sector } AOP < \triangle OPT,$$

$$\therefore \quad \tfrac{1}{2}r^2 \sin\theta < \tfrac{1}{2}r^2\theta < \tfrac{1}{2}r^2 \tan\theta$$

or, $\quad \sin\theta < \theta < \tan\theta$

Cor. If now θ becomes indefinitely small, we can prove,

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1 ; \quad \underset{\theta \to 0}{\text{Lt}} \frac{\tan\theta}{\theta} = 1$$

For, since, $\sin\theta < \theta < \tan\theta$, we get by dividing by $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

i.e., $\theta/\sin\theta$ lies between 1 and $\sec\theta$. But when θ is indefinitely diminished, the limit of $\sec\theta$ is 1 ; hence the limit of

$$\frac{\theta}{\sin\theta} = 1, \text{ that is, } \underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1.$$

Again, by dividing $\sin\theta < \theta < \tan\theta$ by $\tan\theta$,

$$\cos\theta < \frac{\theta}{\tan\theta} < 1. \text{ Hence the limit of } \frac{\tan\theta}{\theta} \text{ is unity ;}$$

i.e., $$\underset{\theta \to 0}{\text{Lt}} \frac{\tan\theta}{\theta} = 1.$$

3·8. To find the value of sin18° geometrically

Let ABC be an isosceles triangle with base angles double the vertical angle at A.

$$\therefore \quad A + 2A + 2A = 180^\circ$$

$$\therefore \quad A = 36^\circ$$

Bisect the angle BAC by AD. So AD is the perpendicular bisector of the base BC.

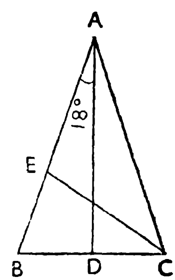


Fig. 62

$$\therefore \quad \angle BAD = 18^\circ$$

$$\therefore \quad \sin 18^\circ = \frac{BD}{BA} = \frac{x}{a},$$

where AB$=a$ and BD$=x$.

From the construction of Euclid IV. 10,

$$AE=BC=2BD=2x$$

and, $$AB.\ BE=AE^2\ ;$$

$$\therefore\quad a\,(a-2x)=(2x)^2$$

$$\therefore\quad 4x^2+2ax-a^2=0$$

$$\therefore\quad x=\frac{-2a\pm\sqrt{20a^2}}{8}=\frac{-1\pm\sqrt{5}}{4}a.$$

Since x is always positive, we must take only the positive value.

$$\therefore\quad \sin\ 18^\circ=\frac{x}{a}=\frac{\sqrt{5}-1}{4}.$$

IV. Harder problems on heights and distances

4-1. As an illustration of the practical application of Trigonometry we discussed some problems on heights and distances in an earlier chapter. Those problems were easier in nature and required a knowledge of trigonmetrical relations involving a right angled triangle only. Here, in this section, we shall discuss problems of more general and necessarily, therefore, of more harder nature. For such problems, a knowledge of the general properties of triangles coupled with geometrical skill is essential.

2-2. Height and distance of an inaccesible object on a horizontal plane

Let AD be the object standing vertically on the horizontal plane which contains B, the position of the observer. It is required to find AD and BD.

Case I. Measure a base line AC directly from B towards the object. The angle of elevation of A is ACD when observed from C. At B, the angle of elevation of A is ABC.

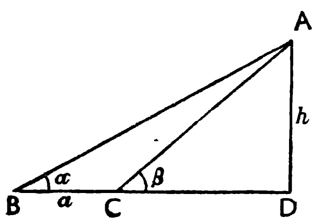


Fig. 63

Let $\angle$ABC$=\alpha$, $\angle$ACD $=\beta$, BC$=a$, AD$=h$, BD$=d$.

Now, $a = \text{BD} - \text{CD}$

$$= h \cot \alpha - h \cot \beta$$

$$= h \left[\frac{\cos \alpha}{\sin \alpha} - \frac{\cos \beta}{\sin \beta}\right]$$

$$= \frac{h \sin (\beta - \alpha)}{\sin \alpha \sin \beta}.$$

$\therefore$ $\mathbf{h = a \sin \alpha \sin \beta \operatorname{cosec} (\beta - \alpha)}$ ......(I)

and $\mathbf{d = BD} = h \cot \alpha = \mathbf{a \cos \alpha \sin \beta \operatorname{cosec} (\beta - \alpha)}$

Case II. If, however, it is not possible to go directly from B towards the object, we measure off a length BC $(= a)$ in any convenient direction. From B and C are observed the angles ABC and ACB respectively.

Let $\angle \text{ABD} = \alpha$, $\angle \text{ABC} = \theta$, $\angle \text{ACB} = \phi$. From $\triangle$ABC,

$$\frac{\text{AB}}{\text{BC}} = \frac{\sin \phi}{\sin \text{BAC}} = \frac{\sin \phi}{\sin (180^\circ - \overline{\theta + \phi})}$$

$\therefore$ $\text{AB} = \text{BC} \sin \phi \operatorname{cosec} (180^\circ - \overline{\theta + \phi})$

$= a \sin \phi \operatorname{cosec} (\theta + \phi)$

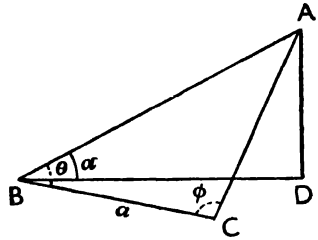


Fig. 66

$\therefore$ $\mathbf{h = AB \sin \alpha = a \sin \alpha \sin \phi \operatorname{cosec} (\theta + \phi)}$ ......(II)

and $\mathbf{d = BD = AB \cos \alpha = a \cos \alpha \sin \phi \operatorname{cosec} (\theta + \phi)}$

Note 1. The expressions (I) and (II) are suitable for logarithmic calculations. For example, from (I)

$$\log h = \log a + \log \sin \alpha + \log \sin \beta + \log \operatorname{cosec} (\beta - \alpha)$$
$$\log d = \log a + \log \cos \alpha + \log \sin \beta + \log \operatorname{cosec} (\beta - \alpha)$$

Note 2. The height of the observer is disregarded so that the angles of elevation are measured from the horizontal plane.

4-3. Distance between two visible but inaccessible objects

Let C and D be the two visible but inaccessible objects. To find the distance CD.

Let A and B be two convenient stations in the same horizontal plane and let the distance between them be c.

At A, measure the angles DAB, CAB and DAC and denote them by α, β, θ respectively. At B, measure the angles DBA and CBA and denote them by γ and δ respectively.

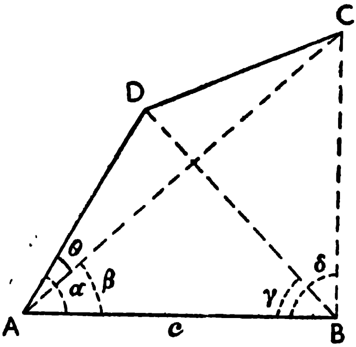


Fig. 67

Now, from $\triangle$DAB,

$$\frac{DA}{\sin\gamma}=\frac{c}{\sin(180^\circ-\overline{\alpha+\gamma})}$$

$$=\frac{c}{\sin(\alpha+\gamma)}$$

$\therefore$ $DA=c\sin\gamma\operatorname{cosec}(\alpha+\gamma)$

Similarly, $CA=c\sin\delta\operatorname{cosec}(\beta+\delta)$

Also, from $\triangle$DAC, $CD^2=DA^2+CA^2-2DA.\ CA\ \cos\theta$, whence CD is determined.

Note. If D, A, B, C all lie in the same plane, it is needless to measure $\angle$DAB ; for, then, $\angle DAB=\angle DAC+\angle CAB=\theta+\beta$.

4-4. Illustrated examples

We give below some more worked out examples of harder nature on heights and distances.

Ex. 1. *A man walking towards a building, on which a flagstaff is fixed, observes the angle subtended by the flagstaff to be greatest when he is at a distance 'd' from the building. If θ be the observed greatest angle, find the length of the flagstaff and the height of the building.* (P. U. 1941)

Let QB denote the building, PQ the flagstaff. Also, let the point A be at a distance d from QB. By the problem, the angle subtended by PQ at A is maximum. To find PQ and QB.

Since A is the point in the horizontal line AB at which PQ subtends a maximum angle, it can be easily proved from

geometry that $\overline{AB}$ touches the circle passing round the $\triangle$ PQA.

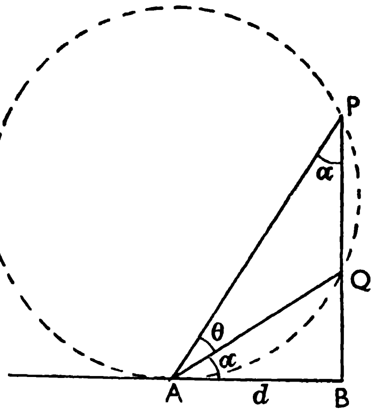


Fig. 68

$$\text{Let } \angle QAB = \angle APQ = \alpha \,;$$

$$\because \quad \angle PAB + \angle APB = 90°,$$

$$\theta + 2\alpha = 90°. \qquad \ldots\ldots\text{(i)}$$

Now, $PQ = PB - QB$

$$= d \tan(\theta+\alpha) - d \tan\alpha$$

$$= d\left[\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)} - \frac{\sin\alpha}{\cos\alpha}\right]$$

$$= d\frac{\sin\theta}{\cos(\theta+\alpha)\cos\alpha}$$

$$= \frac{2d\sin\theta}{\cos(\theta+2\alpha)+\cos\theta}$$

$$= 2d\tan\theta. \qquad (\because \quad \theta+2\alpha=90°)$$

Also, $QB = d\tan\alpha = d\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$.

Ex. 2, *A spherical baloon whose radius is 'r' feet subtends at an observer's eye an angle α, when the angular elevation of its centre is β. Determine the height of the centre of the baloon.*

(C. U. 1953, A. U. 1951)

Let A be the observer's eye, AZ a horizontal line through A. Also, let O be the centre of the spherical baloon.

$$\therefore \quad \angle OAZ = \beta.$$

If AM, AL be the two tangents to the circle from A, $\angle$MAL is the angle subtended at A, by the baloon.

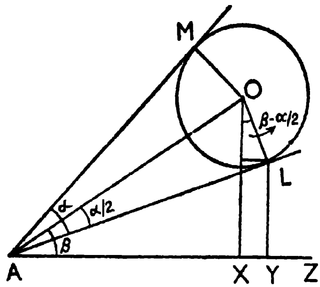


Fig. 69

$\therefore$ $\angle MAL=\alpha$.

$\therefore$ $\angle LAO=\frac{1}{2}\angle MAL=\frac{1}{2}\alpha$.

Draw perpendiculars LY and OX on AZ, and LR on OX.

$\therefore$ height of the centre of the baloon

$=OL \cos LOR+AL \sin LAY$

$=r \cos (\beta-\alpha/2)+OL \cot OAL. \sin LAY$

$=r \cos (\beta-\alpha/2)+r \cot \alpha/2 \sin (\beta-\alpha/2)$

$$=\frac{r}{\sin \alpha/2}\left[\sin\frac{\alpha}{2} \cos (\beta-\alpha/2)+\cos \alpha/2 \sin (\beta-\alpha/2)\right]$$

$$=\frac{r \sin \beta}{\sin \alpha/2}$$

Ex. 3. *The angle of elevation of the top of a hill from a point A is α. After walking a distance 'a' towards the top up a slope inclined to the horizon at an angle θ the angle of elevation is β. Find the height of the hill.*

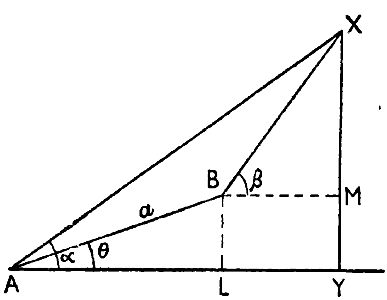


Fig. 70

Let XY be the hill, A and B, the two points of observation.

$\therefore$ $AB=a$, $\angle XAY=\alpha$, $\angle BAY=\theta$, $\angle XBM=\beta$ and BM is the horizontal line through B. Draw BL perp. to AY, Let $XY=x$.

Now, $XM = XY - YM = x - BL = x - a \sin\theta$

and $BM = LY = AY - AL = x \cot\alpha - a\cos\theta$.

$$\therefore \quad \tan\beta = \frac{XM}{BM} = \frac{x - a\sin\theta}{x\cot\alpha - a\cos\theta}$$

$$\therefore \quad x = \frac{a\sin\alpha\sin(\beta-\theta)}{\sin(\beta-\alpha)}.$$

Ex. 4. *The angle of elevation of a light at the top of a distant tower from a point 12 ft. above a lake is 24°55′ and the angle of depression of its reflection in the lake is 35°5′. Find the height of the tower correct to two decimal places, having given log 2 = ·30103, log 3 = ·47712, log 588 = 2·76938, log 589 = 2·77012, L sin 10°10′ = 9·24677.*

Let O be the light at the top of the tower OP, B the observer, C the point of incidence of light ray OC in the lake and CB, therefore, is the reflected ray. Let $OP = h$ ft.

Now, $\angle BCA = \angle OCP = \phi$ (say), from the laws of reflection.

$\therefore \quad \phi = 35°5'$, the angle of depression.

If θ be the angle of elevation of O from B, $\theta = 24°55'$,

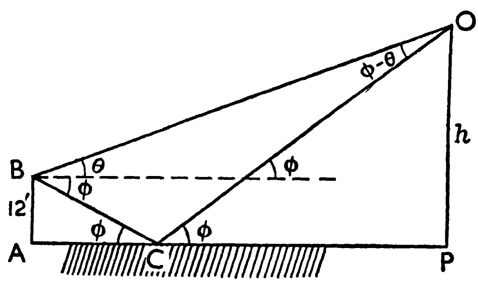


Fig. 71

$$\text{Now,} \quad \frac{CO}{\sin(\theta+\phi)} = \frac{BC}{\sin(\phi-\theta)} = \frac{12}{\sin\phi\sin(\phi-\theta)} \text{ ft.}$$

$$\therefore \quad h = OP = CO\sin\phi = 12\,\frac{\sin(\theta+\phi)}{\sin(\phi-\theta)} = 12\,\frac{\sin 60°}{\sin 10°10'}$$

$$= \frac{6\sqrt{3}}{\sin 10°10'} = \frac{2.\,3^{\frac{3}{2}}}{\sin 10°10'}$$

$\therefore \quad \log h = \log 2 + \frac{3}{2} \log 3 + 10 - L \sin 10°10'$
$= \cdot 30103 + \frac{3}{2} (\cdot 47712) + 10 - 9 \cdot 24677$
$= 1 \cdot 76994.$

$\therefore \quad \log h$ lies between $\log 58 \cdot 8$ and $\log 58 \cdot 9$.

Let $h = 58 \cdot 8 + x$.

Difference for $\cdot 1 = 1 \cdot 77012 - 1 \cdot 76938 = \cdot 00074$

Difference for $x = 1 \cdot 76994 - 1 \cdot 76938 = \cdot 00056$.

$$\therefore \quad \frac{x}{\cdot 1} = \frac{56}{74} = \cdot 75.$$

$\therefore \quad x = \cdot 075 = \cdot 08$ approx.

$\therefore \quad h = 58 \cdot 88$ ft.

Examples A 3

1. Observations to find the height of a mountain are taken at two stations A and B which are at the same height and 6000 ft. apart. The elevation of the top at A is found to be 75° and 60° in order, P being the top. What is the height of the mountain ?

2. At a point on a level plane a tower subtends an angle α, and a man b feet high on its top an angle ϵ ; prove that the height of the tower is

$$\frac{b \sin \alpha \cos (\alpha + \epsilon)}{\sin \epsilon}$$

3. A person walking along a straight road observes that at two consecutive milestones the angles of elevation of a hill in front of him are 30° and 75° ; find the height of the hill.

4. At a point on a level plane a tower subtends an angle α and a flagstaff c ft. in length at the top of the tower subtends an angle β ; show that the height of the tower is

$$c \sin \alpha \operatorname{cosec} \beta \cos (\alpha + \beta)$$

5. The altitude of a rock is observed to be 47° ; after walking 1000 feet towards it up a slope inclined at 32° to the horizon the altitude is 77°. Find the vertical height of the rock above the first point of observation, given $\sin 47° = 0 \cdot 731$

6. A flagstaff is fixed on the top of a wall standing upon a horizontal plane. An observer finds that the angle subtended at a point on this plane by the wall and the flagstaff are α and β respectively. He then walks a distance c directly towards the wall and finds that the flagstaff again subtends an angle β. Find the heights of the wall and the flagstaff.

7. A flagstaff stands on the top of a tower. A man walking along a straight road towards the tower observes that the angle of elevation of the top of the flagstaff is β; after walking a distance a further along the road he notices that the flagstaff subtends its maximum angle α; show that the height of the flagstaff is

$$\frac{2a \sin \alpha \sin \beta}{\cos \beta + \sin (\alpha - \beta)}$$

8. The angular elevation of a tower CD at a place A due south of it is 30°, and at a place B due west of A the elevation is 18°. If AB$=a$, show that the height of the tower is

$$\frac{a}{\sqrt{2+2\sqrt{5}}}$$

9. Two vertical poles whose heights are a and b, subtend the same angle α at a point in the line joining their feet. If they subtend angles β and γ at any point in the horizontal plane at which the line joining their feet subtends a right angle, prove that

$$(a+b)^2 \cot^2 \alpha = a^2 \cot^2 \beta + b^2 \cot^2 \gamma$$

10. The angular elevation of a column viewed from a station P due east of it being θ, and from a station Q due north of the former station being ϕ, show that the height of the tower is

$$\frac{\text{PQ} \sin \theta \sin \phi}{\sqrt{\sin (\theta+\phi) \sin (\theta-\phi)}} \qquad \text{(B. H. U. 1953)}$$

11. At each end of a base line of length $2a$ it is found that the angular altitude of a certain peak is θ, and at the middle point of the base the altitude is ϕ. Prove that the vertical height of the peak above the plane is

$$\frac{a \sin \theta \sin \phi}{\sqrt{\sin (\theta+\phi) \sin (\phi-\theta)}} \qquad \text{(A. U. 1955)}$$

12. A man stands on the top of a wall of height h ft. and observes the elevation of a telegraph post to be α; he then descends from the wall and finds the elevation to be β; show that the height of the post exceeds that of the man by

$$\frac{h \sin \beta \cos \alpha}{\sin (\beta-\alpha)} \text{ ft.}$$

13. A tower is situated on a horizontal plane at a distance a from the base of a hill whose inclination is α. A person on the hill, looking over the tower, can just see a pond, the distance of which from the tower is b. Show that, if the distance of the observer from the foot of the hill be c, the height of the tower is

$$\frac{bc \sin \alpha}{a+b+c \cos \alpha}$$

14. Two land-marks exactly North and South of one another are separated by a river. A person walks due West from one of them a distance a, and then a further distance b; he finds that the angle subtended at his eye by the object in the first case is three times the angle subtended in the second case. Show that the distance between the land-marks is

$$(a+b)\sqrt{\frac{b-2a}{3b+2a}}$$

15. From a house on one side of a road observations are made of the angle subtended by the houses opposite, first from the level of the road and next from a room window at a height c. If these angles be α and β, show that the height of the house h is given by

$$\frac{c^2}{h^2}-\frac{c}{h}=\cot \beta \cot \alpha-\cot^2\alpha$$

16. Two points A and B are observed from two points C and D in the same plane, the distance CD being d. The angles ACD, BCD, ADC, BDC ars respectively α, β, γ, δ and $\alpha+\gamma=\beta+\delta$; show that

$$AB=\frac{d \sin (\alpha-\beta)}{\sin (\alpha+\gamma)}$$

17. In the same horizontal plane there are two inaccessible points P and Q and two stations S and T, at each of which PQ is observed to subtend the angle α, PT subtends the angle β and QS subtends at T the angle γ; show that

$$PQ=\frac{ST\sin\alpha}{\sin(\beta+\gamma-\alpha)} \qquad \text{(B. H. U. 1950)}$$

18. A, B, C are three telegraph posts at equal intervals by the side of a rail road; t, t' are the tangents of the angles which AB and BC subtend at any point P, show that

$$\frac{2}{T}=\frac{1}{t}-\frac{1}{t'}$$

where T is the tangent of the angle which the road makes with BP.

19. A vertical rod is erected in a horizontal rectangular field ABCD. The angular elevation of its top from A, B, C, and D are α, β, γ and δ. Show that

$$\cot^2\alpha-\cot^2\beta=\cot^2\delta-\cot^2\gamma$$

20. The angles of elevation of a bird flying in a horizontal straight line from a fixed point at four successive observations are α, β, γ, δ, the observations being taken at equal intervals of time. Assuming that the speed of the bird is uniform, show that

$$\cot^2\alpha-\cot^2\delta=3(\cot^2\beta-\cot^2\gamma) \qquad \text{(P. U. 1941)}$$

21. A tree standing on a horizontal plane is inclined by θ towards N. At two points due S. and distant a, b, respectively from the foot, the angular elevations of the tree are α and β; show that

$$\tan\theta=(a-b)/(a\cot\beta-b\cot\alpha)$$

22. An observer on a carriage moving with a speed v along a st. road observes in one position that two distant trees are in the same line with him, the line being inclined at an angle to the road. After a time t, he observes that the trees subtend then the greatest angle ϕ; show that the distance between the trees is

$$\frac{2\,vt\sin\theta\sin\phi}{\cos\theta+\cos\phi}$$

23. A spherical time ball of radius r at the top of a tower subtends an angle 2α at a point on the ground from which the elevation of its centre is ϕ ; prove that the height of the centre of the ball above the ground is $r \sin \phi \operatorname{cosec} \alpha$.

24. Two stations due south of a tower, which leans towards the north, are at distances a and b from its foot ; if θ and ϕ are the elevations, prove that its inclination to the horizontal is

$$\cot^{-1}\left(\frac{b \cot \theta - a \cot \phi}{b-a}\right) \qquad \text{(A. U. 1957)}$$

25. The angle of elevation of a cloud from a point x feet above a lake is θ, and the angle of depression of its reflection in the lake is ϕ ; prove that its height is

$$x \frac{\sin(\phi+\theta)}{\sin(\phi-\theta)} \text{ ft.} \qquad \text{(A. U. 1956)}$$

26. A chimney leans towards the N. At equal distances due N. and S. of it on a horizontal plane the elevations of the top are α and β. Prove that the inclination of the chimney to the vertical is

$$\tan^{-1}\frac{\sin(\alpha+\beta)}{2 \sin \alpha \sin \beta} \qquad \text{(A. U. 1954)}$$

27. Determine the height of the mountain if the elevation of its top at an unknown distance from the base is 28°, and at a distance 3 miles 77 yards further off from the mountain along the same line, the elevation is 16°, given

$$\log 1{\cdot}6071 = {\cdot}2060, \ L \sin 12° = 9{\cdot}3179$$

$$L \sin 16° = 9{\cdot}4403, \text{ and } L \sin 28° = 9{\cdot}6716$$

(A. U. 1949)

28. A flagstaff is on the top of a tower which stands on a level plane. At a certain point in the plane the tower subtends an angle α, and the flagstaff an angle β. At another point a ft. nearer the base of the tower the flagstaff again subtends the angle β. Show that the height of the tower is

$$\frac{a \tan \alpha}{1 - \tan \alpha \tan(\alpha+\beta)} \qquad \text{(P. U. 1945)}$$

Miscellaneous examples

1. Express in degrees and minutes and also in grades the vertical angle of an isosceles triangle in which each of the angles at the base is twelve times the vertical angle.

2. On a globe 6 miles diameter an arc 2 fur. 55 yds. is measured; find the radian measure of the angle subtended at the centre of the globe.

If this was taken as the unit of measurement, how would a right angle be represented ?

3. *Prove that*

(i) $\dfrac{\tan x-\cot y}{\tan y-\cot x}=\tan x \cot y$

(ii) $(\sin x+\cos x)(\tan x+\cot x)=\sec x+\operatorname{cosec} x$

4. If $a \cos^2 x+b \sin^2 x=c$, *show that* $\tan x=\pm\sqrt{\dfrac{c-a}{b-c}}$.

5. Which of the following statements is possible and which impossible ?

(i) $\operatorname{cosec} \theta=\dfrac{x^2+y^2}{2xy}$

(ii) $\sec \theta=\dfrac{x^2-y^2}{x^2+y^2}$

(iii) $\sin \theta=\dfrac{x^2+y^2+z^2}{xy+yz+zx}$ x, y, z being unequal real quantities.

6. Show that whatever be the value of θ, the expression $\sin^2(\theta+\alpha)+\sin^2(\theta-\beta)-2\cos(\alpha+\beta)\sin(\theta+\alpha)\sin(\theta+\beta)$ is independent of θ.

7. *Show that* $\sec \phi=\dfrac{2}{\sqrt{2+\sqrt{2+2\cos 4\phi}}}$

8. *Prove that* $\tan\dfrac{A+B}{2}-\tan\dfrac{A-B}{2}=\dfrac{2 \sin B}{\cos A+\cos B}$

9. *If* $\tan \beta=\dfrac{n \sin \alpha \cos \alpha}{1-n \sin^2 \alpha}$, *show that*

$\tan(\alpha-\beta)=(1-n)\tan \alpha$ (P. U. 1941)

10. PQR is a triangle and S is the projection of P on QR produced. If $\angle PQS=30°$, $\angle PQR=45°$ and QR=2 ft., find RS. (C. U. 1951)

11. Solve the equations

(i) $\sin\frac{n+1}{2}\theta=\sin\frac{n+1}{2}\theta+\sin\theta$ (A. U. 1953)

(ii) $\sec\theta-1=(\sqrt{2}-1)\tan\theta$ (B. H. U. 1957)

(iii) $\sqrt{2}\tan\theta\sin\theta+\sqrt{3}=\tan\theta+\sqrt{6}\sin\theta$

12. *If* $\tan\theta\tan\phi=\sqrt{\frac{1-x}{1+x}}$, *show that*

$$(1-x\cos 2\theta)(1-x\cos 2\phi)=1-x^2$$

13. If α and β be the different values of θ which satisfy the equation $a\cos\theta+b\sin\theta=c$, *prove that*

$$\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$$

14. *If* $A+B+C=180°$, *prove that*

$$\cot A+\frac{\cos B}{\sin A\cos C}=\cot B+\frac{\cos A}{\sin B\cos C}$$

15. *If* $A+B+C=180°$, *prove that* the greatest value of $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ is $\frac{1}{8}$

16. *If* $\cos\theta-\sin\theta=\cos\alpha-\sin\alpha$, *prove that*

$$\theta+\frac{\pi}{2}=2n\pi\pm\left(\alpha+\frac{\pi}{4}\right)$$ (B. H. U. 1940)

17. Solve the equations (general values not required)

$$\tan x+\tan y=2$$

and $2\cos x\cos y=1$ (C. U. 1955)

18. Solve $\tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{4x+1}=\tan^{-1}\frac{2}{x^2}$

(Agra. 1947)

19. Solve $\sin^{-1}x+\sin^{-1}y=\frac{2}{3}\pi$

$\cos^{-1}x-\cos^{-1}y=\frac{1}{3}\pi$ (C. U. 1940)

20. *Prove that* $\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}\frac{c-a}{1+ca}$

$$=\tan^{-1}\frac{a^2-b^2}{1+a^2b^2}+\tan^{-1}\frac{b^2-c^2}{1+b^2c^2}+\tan^{-1}\frac{c^2-a^2}{1+c^2a^2}$$

(P. U. 1931)

21. The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Determine the value of the greatest angle.

22. Prove that in a triangle ABC

$$\sin 2A+\sin 2B+\sin 2C=\frac{32\Delta^3}{a^2b^2c^2}$$

23. Show that in a $\triangle ABC$ $\sin A \sin B \sin C=\Delta/2R^2$

24. If the lengths of the perpendiculars from the circum-centre on the sides BC, CA, AB of the $\triangle ABC$ are x, y, z respectively, prove that

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$$

25. If $r : R : r_1 = 2 : 5 : 12$ show that the triangle is right angled.

26. If r and R are the radii of the in-circle and circum-circle of a triangle, prove that

$$8rR\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)$$
$$=2bc+2ca+2ab-a^2-b^2-c^2$$

27. Corresponding to the inequality $a+b>c$ concerning the sides of a triangle, can you prove $\sin A+\sin B>\sin C$?

(C. U. 1950, '54)

28. *If* in the ambiguous case, the angles A and C have two values A_1, A_2 and C_1, C_2 respectively

show that $$\frac{\sin A_1}{\sin C_1}+\frac{\sin A_2}{\sin C_2}=2\cos B$$

29. *If* $b=\sqrt{3}\ c=1$, $A=30°$, solve the triangle.

(C. U. 1951)

30. Walking down a hill inclined to the horizon at an angle θ, a man observes an object in the horizontal plane whose angle of depression is α. Half way down the hill the angle of depression is β. Prove that $\cot\theta=2\cot\alpha-\cot\beta$

31. Eliminate ϕ from

(i) $$\frac{x-a\cos\phi}{a\sin\phi}=\frac{y-b\sin\phi}{-b\cos\phi}=\frac{z}{c}$$

(ii) $\cos 2\phi=a\cos\phi$; $\sin 2\phi=b\sin\phi$

32. Eliminate θ and ϕ from

$$x=a\cos\theta\sec\phi$$
$$y=b\sin\theta\sec\phi$$
$$z=c\tan\phi$$

33. Find the sum to n terms

$$\sqrt{1+\sin\alpha}+\sqrt{1+\sin 2\alpha}+\sqrt{1+\sin 3\alpha}+\ldots\ldots$$

34. *Prove that*

$2\cos\frac{A}{2^n}=\sqrt{2+\sqrt{2+\sqrt{\ldots\sqrt{2+2\cos A}}}}$; the symbol indicating the extraction of the square root being repeated n times.

35. A person walks one mile bearing an angle θ_1 with a fixed direction and then another mile bearing θ_2 with the same direction. Find (a) final distance from the starting point and (b) final bearing. (C. U. 1950)

(**Bearing of a line** : The bearing of a line is the positive acute angle made by the line with the North-South line situated in the same plane).

Answers

Examples 1 (pp. 8-9)

1. 135°, 225°, 3°20′, 85°56′36″, 1°8′45″ nearly, 75°

2. $23\pi/72$, $2\pi/25$, $33\pi/320$, 0·789327 taking $\pi=3{\cdot}1416$

3. $2\pi/5$, $\pi/5$ **4.** $\dfrac{(n-2)\pi}{n}$, $\dfrac{5\pi}{7}$

5. $22\frac{1}{2}°$, $\pi/8$ **6.** $67\frac{1}{2}°$

7. 1·087, 0·913 **8.** 91°, 81°

9. 40°, 60°, 80° **10.** 0·0358 sec. nearly

11. $\pi/5$, $\pi/3$, $7\pi/15$ **12.** $\pi/6$, $\pi/3$, $\pi/2$

13. 75°, 60°, 45°

14. $\dfrac{\pi}{7}$, $\dfrac{2\pi}{7}$, $\dfrac{4\pi}{7}$; $\dfrac{\pi}{21}$, $\dfrac{4\pi}{21}$, $\dfrac{16\pi}{21}$

15. $\pi/8$ **16.** 9 **17.** $\frac{1}{2}$ radian **18.** 13 yds. 2ft. 8 in.

19. 1·53 ft. **20.** 3436·36 ft. nearly **21.** 4 min. $53\frac{1}{3}$ sec.

22. $49\frac{1}{11}$ inches **23.** 33 ft. **24.** 32 : 31

Examples 2 (pp. 19-21)

18. $\frac{13}{12}$, $\frac{5}{12}$ **20.** $\dfrac{p^2-q^2}{p^2+q^2}$ **22.** $\dfrac{p^2-q^2}{p^2+q^2}$, $\dfrac{p^2+q^2}{p^2-q^2}$

23. $\dfrac{2m}{m^2+1}$, $\dfrac{m^2+1}{m^2-1}$ **24.** $\dfrac{2mn}{m^2+n^2}$, $\dfrac{m^2-n^2}{2mn}$

26. $\dfrac{a+b}{\sqrt{2(a^2+b^2)}}$, $\dfrac{a-b}{\sqrt{2(a^2+b^2)}}$ **31.** $(\sin\theta-\cos\theta)^2$

32. $\tan\theta+\dfrac{1}{\tan\theta}$ **34.** $\pm\frac{3}{4}$ or $\pm\sqrt{5/2}$

36. 1 or 2 **39.** $\dfrac{1}{\sqrt{5}}$

Examples 3 (pp. 25)

1. $\sqrt{3}/2$ **2.** 4/3 **3.** $\frac{3}{2}$ **4.** 1 **5.** 5/4

11. 6 **12.** 3 **13.** $16/3\sqrt{3}$ **14.** (i) 60°

(ii) 30° (iii) 60° (iv) 30° (v) 45° **15.** $\theta=52\frac{1}{2}°$, $\phi=7\frac{1}{2}°$

Examples 4 (pp. 36-37)

1. $-\frac{1}{2}$, 1, $-\frac{1}{2}$, $2\sqrt{3}, -1/\sqrt{3}, \frac{1}{2}$

2. $\frac{1}{\sqrt{2}}, -\sqrt{3}, \sqrt{2}$. 3. -4

4. (i) $\pm 30°, \pm 330°$
   (ii) $210°, 330°, -30°, -150°$
   (iii) $30°, 210°$
   (iv) $135°, 315°, -45°, -225°$
   (v) $60°, 120°$
   (vi) $135°, 225°$

5. (i) $\sin\theta$
   (ii) -1
   (iii) $\cot^2\theta$

7. (i) 2 (ii) 2 (iii) o or $\cos\alpha$, according as n is even or odd (iv) 1 (v) 21/26

8. (i) $90°$ (ii) $60°, 300°$ (iii) $60°, 90°, 120°, 240°, 270°, 300°$ (iv) $30°, 150°, 210°, 330°$ (v) $60°, 300°$ (vi) $30°$ $120°, 210°, 300°$ (vii) $0°, 60°, 180°, 300°, 360°$

Examples 5 (pp. 47-48)

1. $\frac{1}{6\sqrt{2}} - \frac{\sqrt{7}}{8}$

2. $-\frac{85}{36}$

Examples 7 (pp. 65-68)

23. $\frac{1}{2}\sqrt{2-\sqrt{2}}, \frac{1}{2}\sqrt{2+\sqrt{2}}$

24. $\frac{1}{16}(\sqrt{5}-1)(6+\sqrt{2})-\frac{1}{8}(\sqrt{3}-1)(\sqrt{5+\sqrt{5}})$, and $\frac{1}{8}(\sqrt{3}+1)(\sqrt{5+\sqrt{5}})+\frac{1}{16}(\sqrt{6}-\sqrt{2})(\sqrt{5}-1)$

36. $\frac{b^2-a^2}{b^2+a^2}$

37. (i) $2n\pi-\pi/4$ and $2n\pi+\pi/4$
   (ii) $2n\pi+5\pi/4$ and $2n\pi+7\pi/4$
   (iii) $2n\pi+3\pi/4$ and $2n\pi+5\pi/4$

38. Both positive

39. No ; $2\sin\frac{A}{2}=\sqrt{1+\sin A}+\sqrt{1-\sin A}$

Examples 8 (pp. 76-77)

1. $n\pi \pm \pi/6$ 2. $n\pi \pm \pi/4$ 3. $2n\pi \pm 2\pi/3$

4. $n\pi + \frac{\pi}{2}$, $n\pi \pm \pi/4$ 5. $n\pi$, $n\pi/2 \pm \pi/12$

6. $n\pi/4$, $\frac{1}{3} n\pi + (-1)^n \pi/18$ 7. $2n\pi \pm \pi/3$, $(2k+1)\pi$

8. $\frac{n\pi}{2} + (-1)^n \pi/12$ 9. $\frac{r\pi}{p+(+1)^r q}$

10. $\frac{(4k+1)\pi}{2(q-p)}, \frac{(4k-1)\pi}{2(q+p)}$

11. $2n\pi$ or $(4n \pm 1)\pi/2$ or $(4n \pm 1)\pi/5$
12. $(2n+1)\pi/2$ or $(2n+1)\pi/4$ $(2n+1)\pi/8$
13. $2n\pi$, $2n\pi \pm 2\pi/3$ 14. $n\pi + (-1)^n \pi/6$, $(4n-1)\pi/2$
15. $(4n+1)\pi/8$ 16. $2n\pi$ or $2n\pi - 2\pi/3$
17. $2n\pi - 7\pi/12$, $2n\pi - \pi/12$ 18. $2n\pi + 7\pi/12$ or $2n\pi + \pi/12$
19. $2n\pi + \pi/4$ 20. $2n\pi - \pi/4$
21. $2n\pi + 5\pi/12$ or $2n\pi - \pi/12$
22. $2n\pi + \pi/2$ or $2n\pi - \beta$ where β is a positive acute angle whose sine is $\frac{3}{5}$
23. $2n\pi + \pi/2$ or $2n\pi - \pi/6$ 24. $2n\pi$, $\frac{1}{6}(4n+1)\pi$
25. $\frac{n\pi}{3}$, $n\pi \pm \tan^{-1}\frac{1}{\sqrt{2}}$ 26. $\frac{1}{6}n\pi$ 27. $n\pi \pm \pi/6$
28. $\frac{1}{2}(n\pi + \alpha)$ where $\tan\alpha = 2$ 29. $n\pi + (-1)^n \pi/6$
30. $n\pi + \pi/4$, $2n\pi + \pi/3$ 31. $\theta = n\pi \pm \pi/4$, $\phi = n\pi \pm \pi/6$
32. $\theta = n\pi \pm \pi/6$, $\phi = n\pi + \pi/3$ 33. $n\pi/3 + \pi/9$
34. $n\pi \pm \pi/4$ or $2n\pi \pm 2\pi/3$ 35. $n\pi + \frac{\alpha}{2}$; $(2n+1)\pi/6 - \frac{\alpha}{6}$
36. $2n\pi - \alpha$, $2n\pi - \pi/2 + \alpha$
37. $2n\pi + \pi/2$, $2n\pi - \pi/2 + 2\alpha$ where α is the radian of 21°48′

Examples 9 (pp. 86-89)

28. (i) 1 (ii) $\frac{-3 \pm \sqrt{17}}{4}$ (iii) o or $\frac{1}{2}$ (iv) o or $\pm\frac{1}{2}$

(v) $\pm\frac{1}{\sqrt{2}}$ (vi) $\pm\frac{1}{\sqrt{2}}$ (vii) $2-\sqrt{3}$

(viii) $\frac{a-b}{1+ab}$ (ix) $\sqrt{3}$ (x) $\frac{1}{6}$ (xi) $\pm\frac{1}{2}$ (xii) 2

29. ± 1 or $(1 \pm \sqrt{2})$ **31.** $(x-y)(1+yz)=(y-z)(1+xy)$

32. (i) 1 (ii) $\dfrac{x+y}{1-xy}$ (iii) $\frac{3}{2}(\sqrt{10}-\sqrt{5})$ (iv) ∞

Examples 11 (pp. 118-122)

41. $24\sqrt{6}$ **42.** 120° **73.** 4, $8\frac{1}{8}$

Examples 12 (pp. 134-137)

1. 113°34′41″ **2.** 55°46′16″ **3.** 104°28′39″ nearly
4. 58°59′33″ **5.** 88°59′40·9″ **6.** 48°11′23″, 58°24′43″, 73°23′54″ **7.** 112°37′2·8″ approx. **8.** 769·8622
9. 152·41 approx. **10.** 89 896 ft. **11.** 172·6436 ft.
12. 27 03·5 **13.** 765·4321, 1035·43 **14.** 76°47′2·2″, 49°12′57·8″ **15.** 22°36′, 12°36′ **16.** 94°42′54″ 25°17′6″ **17.** b=559·63, A=109°39′57″, C=19°38′3″
18. B=71°44′29·5″ C=41°15′30·5″ **19.** 70°53′36″, 49°6′14″
20. 116°33′54″, 26°33′54″ **21.** A=79°35′ or 35°45′ ; C=68°5′ or 111°55′, a=595 81 or 353 95
22. 57°30′11·8″ or its supplement 122°29′48·2′, Yes
23. 32°4535 **24.** not ambiguous, ambiguous, not ambiguous.

Examples 13 (pp. 147-148)

1. 30° **2.** 0 **3.** – 60°, 60° **4.** 0·74 radian nearly
5. 30° **6.** 0°, 60°, 180°, 300, 360° **7.** $\pi/8$, $5\pi/8$
8. $\pi/4$ **10.** $\pi/4$ **11.** 0 **12.** 90° and 46°25′ nearly
13. 22·5° and 112·5° **16.** 38°10′ nearly **18.** 5 **19.** The graphs will touch each other at the origin.

Examples 14 (pp. 156-158)

1. 69·28 ft. nearly **2.** 86 6 ft. nearly **3.** 30°
4. $259\frac{1}{2}$ ft. nearly **5.** 22·5 ft 39 97 ft. **6.** 51 ft. 81 ft.
7. 283 yds. nearly **8.** $a/\sqrt{2}$ **9.** 68·3 ft. nearly.
10. 0·86 mile approx. **11.** 236 ft., 136 ft. **12.** 519·6 ft.
13. $130\frac{2}{3}$ ft. **14.** 69-28 ft. **15.** 2·73 miles approx.

Examples A 1 (pp. 161-162)

1. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ 2. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$ 3. $b^2=a^2(2-a^2)$

4. $x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{4}{3}}=1$ 5. $(mr-nq)^2+(np-lr)^2=(lq-mp)^2$

6. $(a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=2$ 7. $\frac{ax}{l}-\frac{by}{m}=\frac{c^2}{\sqrt{m^2+l^2}}$

8. $a+b=2ab$ 9. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\frac{c}{2}$

10. $pq(pq-4)=(p+q)^2\tan^2 r$ 11. $\frac{(a+b)(m+n)}{2mn}$

Examples A 2 (pp. 166)

1. $\frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}\sin\frac{n+1}{2}\alpha$ 2. $\frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}\cos\frac{n+1}{2}\alpha$

3. $\frac{\sin\{\alpha+\frac{1}{2}(n-1)(\alpha+\pi)\}\sin\frac{1}{2}n(\alpha+\pi)}{\sin\frac{1}{2}(\alpha+\pi)}$

4. $-\sec\frac{\alpha}{2}\sin\frac{n}{2}(\pi+\alpha)\cos\frac{n+1}{2}(\pi+\alpha)$

5. $\frac{\sin\frac{n(\beta+\pi)}{2}}{\sin\frac{\beta+\pi}{2}}\sin\left\{\alpha+\frac{(n-1)(\beta+\pi)}{2}\right\}$.

6. $\frac{n}{2}+\frac{\sin n\theta}{2\sin\theta}\cos(n+1)\theta.$ 7. $\frac{1}{4}\left(\frac{3\sin^2 n\alpha}{\sin\alpha}-\frac{\sin^2 3n\alpha}{\sin 3\alpha}\right)$

8. $\frac{3\sin n\alpha\cos n\alpha}{4\sin\alpha}+\frac{\sin 3n\alpha\cos 3n\alpha}{4\sin 3\alpha}$

9. $(-1)^{n-1}\frac{\sin n\theta\sin(n+1)\theta}{2\cos\theta}$

10. $(-1)^n\frac{\sin n\theta\sin(n+1)\theta}{2\cos\theta}$

11. $\cos\{\theta+\frac{1}{2}(n-1)(\theta+\frac{1}{2}\pi)\}\frac{\sin\frac{1}{2}n(\theta+\frac{1}{2}\pi)}{\sin\frac{1}{2}(\theta+\frac{1}{2}\pi)}$

12. $\dfrac{(n+1)\cos\{\alpha+(n-1)\beta\}-\cos(\alpha-\beta)-n\cos(\alpha+n\beta)}{2(1-\cos\beta)}$

13. $\dfrac{(n+1)\sin\{\alpha+(n-1)\beta\}-\sin(\alpha-\beta)-n\sin(\alpha+n\beta)}{2(1-\cos\beta)}$

14. $\dfrac{2}{4\sin\alpha}\{(n+1)\sin 2\alpha-\sin 2(n+1)\alpha\}$

15. $\cot\alpha-2^n\cot 2^n\alpha$ 16. $\frac{1}{2}(\tan 3^n x-\tan x)$

17. $\tan^{-1}\dfrac{n}{2+n}$

Examples A 3 (pp. 182-186)

1. $9000\sqrt{2}$ ft. 3. 3606·24 ft. nearly 5. 1034 ft. nearly.

6. $\dfrac{c\sin\alpha\cos(\alpha+\beta)}{\cos(2\alpha+\beta)}$, $\dfrac{c\sin\beta}{\cos(2\alpha+\beta)}$ 27. 10,000 ft.

Miscellaneous Examples (pp. 187-190)

1. 7°12′ ; 8 grades 2. ·09375, 16·7552

5. possible, impossible, impossible 10. $2/(\sqrt{3}-1)$ ft.

11. $2k\pi$, $\dfrac{4k\pi}{1\pm n}$ 17. $x=y=45°$ 18. $-\frac{2}{3}$ or 3

19. $x=\frac{1}{3}$, $y=1$ 20. 120° 29. $a=1$, B=120°, C=30°

31 (i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ (ii) $b(b-a)=2$

32. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ 33. $\dfrac{1}{\sqrt{2}}\operatorname{cosec}\dfrac{\alpha}{4}\sin\dfrac{n\alpha}{n}\sin\left\{\dfrac{\pi}{4}+(n+1)\dfrac{\alpha}{4}\right\}$

35. $2(1+\cos\overline{\theta_1-\theta_2})$, $\dfrac{\theta_1+\theta_2}{2}$

TABLES OF LOGARITHMS OF NUMBERS,

NATURAL SINES,

NATURAL TANGENTS,

LOGARITHMIC SINES,

LOGARITHMIC TANGENTS ETC.

TABLE I
LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	7	9	1	2	3	4	5	6	7	8	9
											Mean Difference								
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743	42	83	125	166	208	248	290	331	373
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555	38	76	114	152	190	227	265	302	340
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059	35	70	105	140	175	209	243	278	313
13	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301	32	65	97	129	162	193	225	258	290
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319	30	60	90	120	150	180	210	240	270
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140	28	56	84	112	140	168	196	224	252
16	20412	20683	20952	21219	21484	21748	22011	22272	22531	22789	26	53	79	105	132	158	184	210	237
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285	25	50	74	99	124	149	174	199	223
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646	23	47	70	94	117	141	164	188	211
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885	22	45	67	89	111	134	156	178	201
20	30803	30320	30535	30750	30963	31175	31387	31597	31806	32015	21	42	64	85	106	127	148	170	191
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044	20	40	61	81	101	121	141	162	182
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984	19	39	58	77	97	116	135	154	174
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840	19	37	56	74	93	111	130	148	167
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620	18	36	53	71	89	107	124	142	160
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330	17	34	51	68	85	102	119	136	153
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975	16	33	49	66	82	98	115	131	148
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560	16	32	47	63	79	94	111	126	142
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090	15	30	46	61	76	91	106	122	137
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567	15	29	44	59	74	88	103	118	132

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996	14	29	43	57	72	86	100	114	129
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379	14	28	42	55	69	83	97	110	125
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720	13	27	40	54	67	80	94	107	121
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020	13	26	39	52	65	78	91	104	117
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283	13	25	38	50	63	76	88	101	113
35	54437	54531	54654	54777	54900	55023	55145	55267	55388	55509	12	24	37	49	61	73	86	98	100
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703	12	24	36	48	60	71	83	95	107
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864	12	23	35	46	58	70	81	93	104
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995	11	23	34	45	57	68	79	90	102
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097	11	22	33	44	55	66	77	88	99
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172	11	21	32	43	54	64	75	86	97
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221	10	21	31	42	52	63	73	84	94
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246	10	20	31	41	51	61	71	82	92
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246	10	20	30	40	50	60	70	80	90
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225	10	20	29	39	49	59	68	78	88
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181	10	19	29	38	48	57	67	76	86
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117	9	19	28	37	47	56	65	75	84
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034	9	18	27	37	46	55	64	73	82
48	68124	68215	68305	68395	68485	68574	68664	68753	98842	68931	9	18	27	36	45	53	62	71	80
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810	9	18	26	35	44	53	61	70	79
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672	9	17	26	34	43	52	60	69	77
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517	8	17	25	34	42	51	59	67	76
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346	8	17	25	33	42	50	58	66	75
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159	8	16	24	32	41	49	57	65	73
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957	8	16	24	32	40	48	56	64	72
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	8	16	23	31	39	47	55	62	70
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	8	15	23	31	39	46	54	62	69
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	8	15	23	30	38	45	[illegible]	60	68
58	76340	76418	76492	76567	76641	76716	76790	76864	76938	77012	7	15	22	30	37	45	52	59	67
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	7	15	22	29	37	44	51	58	66
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	7	14	22	29	36	43	50	57	65
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	7	14	21	28	35	42	49	56	64
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	7	14	21	28	35	42	49	56	63
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	7	14	21	27	34	41	48	55	62
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224	7	13	20	27	34	40	47	54	60
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889	7	13	20	26	33	40	46	53	59
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543	7	13	20	26	33	39	46	52	59
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187	6	13	19	26	32	39	45	52	58
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822	6	13	19	25	32	38	44	50	57
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448	6	13	19	25	31	37	44	50	56
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065	6	12	18	25	31	37	43	49	55
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673	6	12	18	24	30	36	42	49	55
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273	6	12	18	24	30	36	42	48	54
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864	6	12	18	24	30	35	41	47	53
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448	6	12	18	23	29	35	41	47	52

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024	6	12	17	23	29	35	40	46	52
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593	6	11	17	23	29	34	40	46	51
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154	6	11	17	22	28	34	39	45	50
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708	6	11	17	22	28	33	39	44	50
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255	5	11	16	22	27	33	38	44	49
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795	5	11	16	22	27	32	38	43	49
81	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328	5	11	16	21	27	32	37	43	48
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855	5	11	16	21	26	32	37	42	47
83	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376	5	10	16	21	26	31	36	42	47
84	92428	92480	92531	92583	92634	98686	92737	92788	92840	92891	5	10	15	21	26	31	36	41	45
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399	5	10	15	20	25	30	36	41	46
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902	5	10	15	20	25	30	35	40	45
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399	5	10	15	20	25	30	35	40	45
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890	5	10	15	20	25	29	34	39	44
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376	5	10	15	19	24	29	34	39	44
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856	5	10	14	19	24	29	34	38	43
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332	5	9	14	19	24	29	33	38	43
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802	5	9	14	19	24	28	33	38	42
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267	5	9	14	19	23	28	33	37	42
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727	5	9	14	18	23	28	32	37	41
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182	5	9	14	18	23	27	32	36	41
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632	5	9	14	18	23	27	32	36	41
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078	4	9	13	18	22	27	31	36	40
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520	4	9	13	18	22	26	31	35	40
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957	4	9	13	17	22	26	30	35	39

TABLE II.
NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		Mean Differences 1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	0·00000	0·00291	0·00582	0·00873	0·01164	0·01454	0·01745	89°	29	58	87	116	145	175	204	233	262
1°	·01745	·02036	·02327	·02618	·02908	·03199	·03490	88°	29	58	87	116	145	175	204	233	262
2°	·03490	·03781	·04071	·04362	·04653	·04943	·05234	87°	29	58	87	116	145	175	204	233	262
3°	·05234	·05524	·05814	·06105	·06395	·06685	·06976	86°	29	58	87	116	145	174	203	232	261
4°	·06976	·07266	·07556	·07846	·08136	·08426	·08716	85°	29	58	87	116	145	174	203	232	261
5°	0·08716	0·09005	0·09295	0·09585	0·09874	0·10164	0·10453	84°	29	58	87	116	145	174	203	232	261
6°	·10453	·10742	·11031	·11323	·11609	·11898	·12187	83°	29	58	87	116	145	174	203	232	261
7°	·12187	·12476	·12764	·13053	·13341	·13629	·13917	82°	29	58	87	116	145	173	202	231	260
8°	·13917	·14205	·14493	·14781	·15069	·15356	·15643	81°	29	58	86	115	144	173	202	230	259
9°	·15643	·15931	·16218	·16505	·16792	·17078	·17365	80°	29	58	86	115	144	172	201	230	258
10°	0·17365	0·17651	0·17937	0·18224	0·18509	0·18795	0·19081	79°	29	57	86	115	144	172	201	229	258
11°	·19081	·12366	·19652	·19937	·20222	·20507	·20791	78°	29	57	86	114	143	171	200	228	257
12°	·20791	·21076	·21360	·21644	.21928	·22212	·22495	77°	28	57	85	114	142	170	199	227	256
13°	·22495	·22778	·23062	·23345	·23627	·23910	·24192	76°	28	57	85	113	141	170	198	226	255
14°	·24192	·24474	·24756	·25038	·25320	·25601	·25882	75°	28	56	85	113	141	169	197	226	254
15°	0·25882	0·26163	0·26443	0·26724	0·27004	0·27284	0·27564	74°	28	56	84	112	140	168	196	224	252
16°	·27564	·27843	·28123	·28402	·28680	·28959	·29237	73°	28	56	84	112	140	167	195	223	251
17°	·29237	·29515	·29793	·30071	·30348	·30625	·30902	72°	28	56	83	111	139	166	194	222	250
18°	·30902	·31178	·31458	·31730	·32006	·32282	·32557	71°	28	55	83	110	138	166	193	221	248
19°	·32557	·32832	·33106	·33381	·33655	·33929	·34202	70°	27	55	82	110	137	164	192	219	247

	60′	50′	04′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
20°	0·34202	0·34475	0·34748	0·35021	0·35293	0·35565	0·35837	69°	27	55	82	109	137	164	191	218	246
21°	·35837	·36108	·36379	·36650	·36921	·37191	·37461	68°	27	54	81	108	136	163	190	217	244
22°	·37462	·37730	·37999	·38268	·38537	·38805	·39073	67°	27	54	81	108	135	161	188	215	242
23°	·39073	·39341	·39608	·39875	·40142	·40408	·40674	66°	27	53	80	107	134	160	187	214	240
24°	·40674	·40939	·41204	·41469	·41734	·41998	·42262	65°	27	53	80	106	133	159	186	212	238
25°	0·42262	0·42525	0·42788	0·43051	0·43313	0·43575	0·43837	64°	26	52	79	105	131	157	184	210	236
26°	·43837	·44098	·44359	·44620	·44880	·45140	·45399	63°	26	52	78	104	130	156	182	208	234
27°	·45399	·45658	·45917	·46175	·46433	·46690	·46947	62°	26	52	77	103	129	155	181	206	232
28°	·46947	·47204	·47460	·47716	·47971	·48226	·48481	61°	26	51	77	102	128	154	179	204	230
29°	·48481	·48735	·48989	·49242	·49495	·49748	·50000	60°	25	51	76	101	127	152	177	202	228
30°	0·50000	0·50252	0·50503	0·50754	0·51004	0·51254	0·51504	59°	25	50	75	100	125	150	175	200	225
31°	·51504	·51753	·52002	·52250	·52498	·52745	·52992	58°	25	50	74	99	124	149	174	198	223
32°	·52992	·53238	·53484	·53730	·53975	·54220	·54464	57°	25	49	74	98	123	147	172	196	221
33°	·54464	·54708	·54951	·55194	·55436	·55678	·55919	56°	24	49	73	97	122	146	170	194	219
34°	·55919	·56160	·56401	·56641	·56880	·57119	·57358	55°	24	48	72	96	120	144	168	292	216
35°	0·57358	0·57596	0·57833	0·58070	0·58307	0·58543	0·58779	54°	24	47	71	95	119	142	166	190	213
36°	·58779	·59014	·59248	·59482	·59716	·59949	·60182	53°	23	47	70	94	117	140	164	187	211
37°	·60182	·60414	·60645	·60876	·61107	·61337	·61566	52°	23	46	70	92	116	139	162	185	208
38°	·61566	·61695	·62024	·62251	·62479	·62706	·62932	51°	23	46	68	91	114	137	159	182	205
39°	·62932	·63158	·63383	·63608	·63832	·64056	·64279	50°	22	45	67	90	112	135	157	179	202
40°	0·64279	0·64501	0·64723	0·64945	0·65166	0·65386	0·65606	49°	22	44	66	88	111	133	155	177	199
41°	·65606	·65825	·66044	·66262	·66480	·66697	·66913	48°	22	44	65	87	109	131	153	174	196
42°	·66913	·67129	·67344	·67559	·67773	·67987	·68200	47°	21	43	64	86	107	129	150	172	193
43°	·68200	·68412	·68624	·68835	·69046	·69256	·69466	46°	21	42	63	84	106	127	148	169	190
44°	·69466	·69675	·69883	·70091	·70298	·70505	·70711	45°	21	42	62	83	104	124	145	166	187

NATURAL COSINES

NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	0·70711	0·70916	0·71121	0·71325	0·71529	0·71732	0·71934	44°	20	41	61	82	102	122	143	163	184
46°	·71934	·72136	·72337	·72537	·72737	·72937	·73135	43°	20	40	60	80	100	120	140	160	180
47°	·73135	·73333	·73531	·73728	·73924	·74120	·74314	42°	20	39	59	78	98	118	138	157	177
48°	·74314	·74509	·74703	·74896	·75088	·75280	·75471	41°	19	39	58	77	96	116	135	154	173
49°	·75471	·75661	·75851	·76041	·76229	·76417	·76604	40°	19	38	57	76	95	113	132	151	170
50°	0·76604	0·76791	0·76977	0·77162	0 77347	0·77531	0 77715	39°	19	37	56	74	93	111	130	148	167
51°	·77715	·77897	·78079	·78261	·78442	·78622	·78801	38°	18	36	54	72	91	109	127	145	163
52°	·78801	·78980	·79158	·72335	·79512	79688	·79864	37°	18	35	53	71	89	106	124	142	159
53°	·79864	·80038	·80212	·80386	·80[illegible]58	·80730	·80902	36°	17	35	52	69	87	104	121	138	156
54°	·80902	·81072	·71242	·81412	·81580	·81748	·81915	35°	17	34	51	68	85	101	118	135	152
55°	0·81915	0·82082	0·82248	0·82413	0 82577	0·82741	0·82904	34°	16	33	49	66	82	99	115	132	148
56°	·82904	·83066	·83228	·83389	·83549	·83708	·83867	33°	16	32	48	64	80	96	112	128	144
57°	·83867	·84025	·84182	·84339	·84495	·84650	·84805	32°	16	31	47	63	78	94	110	125	141
58°	·84805	·84959	·85112	·85264	·85416	·85567	·85717	31°	15	30	46	61	76	91	106	122	137
59°	·85717	·85866	·86015	·86163	·86310	·86457	·86603	30°	15	30	44	59	74	89	103	118	133
60°	0·86603	0·86748	0·86892	0·87036	0·87178	0 87321	0·87462	29°	14	29	43	57	72	86	100	114	129
61°	·87462	.87603	·87743	·87882	·88020	·88158	·88295	28°	14	28	42	55	69	83	97	111	125
62°	·88295	·88431	·88566	·88701	·88835	·88968	·89101	27°	13	27	40	54	67	81	94	108	121
63°	·89101	·89232	·89363	·89493	·89623	·89752	·89879	26°	13	26	39	52	65	78	91	104	117
64°	·89879	·90007	·90133	·90259	·90389	·90507	·90631	25°	13	25	38	50	63	75	88	100	113

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
65°	0·90631	0·90753	0·90875	0·90996	0·91116	0·91236	0·91355	24°	12	24	36	48	60	72	84	96	108
66°	·91355	·91472	·91590	·91706	·91822	·91936	·92050	23°	12	23	35	46	58	70	81	93	104
67°	·92050	·92164	·92276	·92388	·92499	·92609	·92718	22°	11	22	33	45	56	67	78	89	100
68°	·92718	·92827	·92935	·93042	·93148	·93253	·93358	21°	11	21	32	43	53	64	75	85	96
69°	·93358	·93462	·93565	·93667	·93769	·93869	·93969	20°	10	20	31	41	51	61	71	81	92
70°	0 93969	0·94068	C·94167	0·94264	0·94361	0·94457	0·94552	19°	10	19	29	39	49	58	68	78	87
71°	·94552	·94646	·94740	·94832	·94924	·95015	·95106	18°	9	18	28	37	46	55	64	74	83
72°	·95106	·95195	·95284	·95372	·95459	·95545	·95630	17°	9	18	26	35	44	52	61	70	79
73°	·95630	·95715	·95799	·95882	·95964	·96046	·96126	16°	8	17	25	33	41	50	58	66	74
74°	·96126	·96206	·96285	9·6363	·96440	·96517	·96593	15°	8	16	23	31	39	47	54	62	70
75°	0·96593	0·96667	0·96742	0·96815	0 96887	0·96959	0·97030	14°	7	15	22	29	36	44	51	58	65
76°	·97030	·97100	·97169	·97237	·97304	·97371	·97437	13°	7	14	20	27	34	41	47	54	61
77°	·97437	·97502	·97566	·97630	·97692	·97754	·97815	12°	6	13	19	25	32	38	44	50	57
78°	·97815	·97875	·97934	·97992	·98050	·98107	·98163	11°	6	12	17	23	29	35	41	46	52
79°	·98163	·98218	·98272	·98325	·98378	·98430	·98481	10°	5	11	16	21	27	32	37	42	48
80°	0·98481	0·98531	0·98580	0·98629	0·98676	0·98723	0·98769	9°	5	10	14	19	24	29	34	38	43
81°	·98769	·98814	·98858	·98902	·98944	·98986	·99027	8°	4	9	13	17	22	26	30	34	39
82°	·99027	·99067	·99106	·99144	·99182	·99219	·99255	7°	4	8	11	15	19	23	27	30	34
83°	·99255	·99290	·99324	·99357	·99390	·99421	·99452	6°	3	7	10	13	17	20	23	26	30
84°	·99452	·99482	·99511	·99540	·99567	·99594	·99619	5°	3	6	8	11	14	17	20	22	25
85°	0·99619	0·99644	0·99668	0·99692	0·99714	0·99736	0·99756	4°	2	5	7	9	12	14	16	18	21
86°	·99756	·99776	·99795	·99813	·99831	·99847	·99863	3°	2	4	5	7	9	11	13	14	16
87°	·99863	·99878	·99892	·99905	·99917	·99929	·99939	2°	1	3	4	5	7	8	9	10	12
88°	·99939	·99949	·99958	·99966	·99973	·99979	·99985	1°									
89°	·99985	·99989	·99993	·99996	·99998	1 00000	1·00000	0°									
90°	1·00000																
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COSINES

TABLE III
NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
0°	0·00000	0·00291	0·00582	0·00873	0·01164	0·01455	0·01746	89°	29	58	87	116	146	175	204	233	262
1°	·01746	·02037	·02328	·02619	·02910	·03201	·03492	88°	29	58	87	116	146	175	204	233	262
2°	·03492	·03783	·04075	·04366	·04658	·04949	·05241	87°	29	58	87	116	146	175	204	233	262
3°	·05241	·05533	·05824	·06116	·06408	·06700	·06993	86°	29	58	88	117	146	175	204	234	263
4°	·06993	·07285	·07870	·07870	·08163	·08456	·08749	85°	29	58	88	117	146	175	204	234	263
5°	0·08749	0·09042	0·09335	0·09629	0·09923	0·10216	0·10510	84°	29	59	88	118	147	176	206	235	265
6°	·10510	·10805	·11099	·11394	·11688	·11983	·12278	83°	29	59	88	118	147	176	206	235	265
7°	·12278	·12574	·12869	·13165	·13461	·13758	·14054	82°	30	59	89	118	148	178	207	237	266
8°	·14054	·14351	·14648	·14945	·15243	·15540	·15838	81°	30	59	89	119	149	178	208	238	267
9°	·15838	·16137	·16435	·16734	·17033	·17333	·17633	80°	30	60	90	120	150	179	209	239	269
10°	0·17633	0·17933	0·18233	0·18534	0·18835	0·19136	0·19438	79°	30	60	90	120	151	181	211	241	271
11°	·19438	·19740	·20042	·20345	·20648	·20952	·21256	78°	30	61	91	121	152	182	212	242	273
12°	·21256	·21560	·21864	·22169	·22475	·22781	·23087	77°	31	61	92	122	153	183	214	244	275
13°	·23087	·23393	·23700	·24008	·24316	·24624	·24933	76°	31	62	92	123	154	185	216	246	277
14°	·24933	·25242	·25552	·25862	·26172	·26483	·26795	75°	31	62	93	124	155	186	217	248	279
15°	0·26795	0·27107	0·27419	0·27732	0·28046	·28360	·28675	74°	31	63	94	125	157	188	219	250	282
16°	·28675	·28990	·29305	·29621	·29938	·30255	·30573	73°	32	63	95	126	158	190	221	253	285
17°	·30573	·30891	·31210	·31530	·31840	·32171	·32492	72°	32	64	96	128	160	192	224	256	288
18°	·32492	·32814	·33136	·33460	·33783	·34108	·34433	71°	32	65	97	129	162	194	226	259	291
19°	·34433	·34758	·35085	·35412	·35740	·36068	·36397	70°	33	65	98	131	164	196	229	262	294

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
20°	0·36397	0·36727	0·37057	0·37388	0·37720	0·38053	0·38386	69°	33	66	100	133	165	199	232	265	298
21°	·38386	·38721	·39055	·39391	·39727	·40065	·40403	68°	34	67	101	134	168	2 2	236	269	302
22°	·40403	·40741	·41081	·41421	·41763	·42105	·42447	67°	34	68	102	136	170	205	239	273	305
23°	·42447	·42791	·43136	·43381	·43828	·44175	·44523	66°	35	69	104	138	173	208	242	277	311
24°	·44523	·44872	·45222	·45573	·45924	·46277	·46631	65°	35	70	105	140	176	211	245	281	316
25°	0·46631	0·46985	0 47341	0·47698	0·48055	0·48414	0·48773	64°	36	71	107	143	179	214	250	285	321
26°	·48773	·49134	·49495	·49858	·50222	·50587	·50953	63°	36	73	109	145	182	218	254	291	327
27°	·50953	·51320	·51688	·52057	·52427	·52798	·53171	62°	37	74	111	148	185	222	259	296	333
28°	·53171	·53545	·53920	·54296	·54673	·55051	·55431	61°	38	75	113	151	189	225	264	302	339
29°	·55431	·65812	·56194	·56577	·56962	·57348	·57735	60°	38	77	115	154	192	230	269	307	346
30°	0·57735	0·58124	0 58513	0 58905	0 59297	0·59691	0·60086	59°	39	78	118	157	196	235	274	313	353
31°	·60086	·60483	·60881	·61280	·61681	·62083	·62487	58°	40	80	120	160	200	240	280	320	360
32°	·62487	·62892	·63299	·63707	·64117	·64528	·64941	57°	41	82	123	164	205	245	285	327	368
33°	·64947	·65355	·65771	·66189	·66508	·67028	·67451	56°	42	84	126	167	209	251	293	334	376
34°	·67451	·67875	·68301	·68728	·69157	·69588	·70021	55°	43	86	128	171	214	257	300	343	385
35°	0·70021	0·70455	0·70891	0·71329	0·71769	0·72218	0·72654	54°	44	88	132	176	220	263	307	351	395
36°	·72654	·73100	·73547	·73996	·74447	·74900	·75355	53°	45	90	135	180	225	270	315	360	405
37°	·75355	·75812	·76272	·76733	·77196	·77661	·78129	52°	46	92	139	185	231	277	324	370	416
38°	·78129	·78598	·79070	·79544	·80020	·80498	·80978	51°	48	95	143	190	238	285	333	380	428
39°	·80978	·81461	·81946	·82434	·82923	·83415	·83910	50°	49	98	147	196	245	293	342	391	440
40°	0·83910	0 84407	0 84906	0·84408	0 85912	0·86419	0·86929	49°	50	101	151	201	252	302	352	402	453
41°	·86929	·87441	·87954	·88473	·88992	·89515	·90040	48°	52	104	156	208	260	311	363	415	467
42°	·90040	·90569	·91099	·91633	·92170	·92709	·93252	47°	54	107	161	214	268	321	375	429	482
43°	·93252	·93797	·94345	·94896	·95451	·96008	·96569	46°	55	111	166	221	277	332	387	442	498
44°	·96569	·97133	·97700	98270	·98843	·99420	1·00000	45°	57	114	172	229	286	353	400	457	515

NATURAL COTANGENTS

NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	1·00000	1·00583	1·01170	1·01761	1·02355	1·02952	1·03553	44°	59	118	178	237	296	355	414	474	533
46°	·03553	·04158	·04766	·05378	·05994	·06613	·07237	43°	61	123	184	246	307	368	430	491	553
47°	·07237	·07864	·08496	·09131	·09770	·10414	·11061	42°	64	127	191	255	319	382	446	510	573
48°	·11061	·11713	·12369	·13029	·13694	·14363	·15037	41°	66	132	199	265	332	397	463	530	596
49°	·15037	·15715	·16398	·17085	·17777	·18474	·19175	40°	69	138	207	276	345	413	482	552	620
50°	1·19175	1·19882	1·20593	1·21310	1·22031	1·22758	1·23490	39°	72	144	216	288	360	431	503	575	647
51°	·23490	·24227	·24969	·25717	·26471	·27230	·27994	38°	75	150	225	300	376	451	526	601	676
52°	·27994	·28764	·29541	·30323	·31110	·31904	·32704	37°	78	157	235	314	392	471	549	628	707
53°	·32704	·33511	·34323	·35142	·35968	·36800	·37638	36°	82	164	247	329	411	493	576	658	740
54°	·37638	·38484	·39336	·40195	·41061	·41934	·42815	35°	86	172	259	345	431	517	603	690	776
55°	1·42815	1·43703	1·44598	1·45501	1·46411	1·47330	1·48256	34°	91	181	272	363	453	544	634	725	816
56°	·48256	·49190	·50133	·51084	·52043	·53010	·53987	33°	96	191	287	382	478	573	669	764	860
57°	·53987	·54972	·55966	·56969	·57981	·59002	·60033	32°	101	201	302	403	504	604	705	806	907
58°	·60033	·61074	·62125	·63185	·64256	·65337	·66428	31°	107	213	320	426	533	639	746	852	959
59°	·66428	·67530	·68643	·69766	·70901	·72047	·73205	30°	113	226	119	451	565	677	790	903	1016
60°	1·7321	1·7437	1·7556	1·7675	1·7796	1·7917	1·8040	29°	12	24	36	48	60	72	84	96	108
61°	1·8040	1·8165	1·8291	1·8418	1·8676	1·8676	1·8807	28°	13	25	38	51	64	77	89	102	115
62°	1·8807	1·8940	1·9074	1·9210	1·9347	1·9486	1·9626	27°	14	27	41	54	68	82	95	109	122
63°	1·9626	1·9768	1·9912	2·0057	2·0204	2·0353	2·0503	26°	15	29	44	58	73	88	102	117	131
64°	2·0503	2·0655	2·0809	2·0965	2·1123	2·1283	2·1445	25°	16	31	47	63	79	94	110	126	141

65°	2·1445	2·1609	2·1775	2.1943	2.2113	2.2286	2·2460	24°	17	34	51	68	85	101	118	135	152
66°	2·2460	2·2637	2·2817	2·2998	2·3183	2·3369	2·3559	23°	18	37	55	73	92	110	128	146	165
67°	2·3559	2·3750	2·3945	2·4142	2·4342	2 4545	2·4751	22°	20	40	60	80	100	119	139	159	179
68°	2·4751	2·4960	2·5172	2·5386	2·5605	2·5826	2 6051	21°	22	43	65	87	109	130	152	174	195
69°	2·6051	2·6279	2·6511	2·6746	2·6985	2·7228	2·7475	20°	24	47	76	95	119	142	160	190	213
70°	2·7475	2·7725	2·7980	2·8239	2·8502	2·8770	2·9042	19°	26	52	78	104	131	157	183	209	235
71°	2·9042	2·9319	2·9600	2·9887	3·0178	3 0475	3·0777	18°	29	58	87	116	145	174	202	231	260
72°	3·0777	3·1084	3·1397	3·1716	3·2041	3·2371	3·2709	17°	32	64	97	129	161	193	225	258	290
73°	3·2709	3·3052	3·3402	3·3759	3·4124	3·4495	3·4874	16°	36	72	108	144	181	216	253	289	325
74°	3·4874	3·5261	3·5656	3·6059	3·6470	3 6891	3·7321	15°	41	81	122	163	204	244	285	326	366
75°	3·7321	3·7760	3·8208	3·8667	3·9136	3·9617	4·0108	14°	46	93	139	185	232	278	325	371	481
76°	4·0108	4·0611	4 1126	4·1653	4·2193	4 2747	4·3325	13°	53	107	160	214	267	320	374	424	481
77°	4·3315	4·3897	4·4494	4·5107	4·5736	4·6382	4·7046	12°	62	124	186	248	311	373	435	497	559
78°	4·7046	4·7729	4·8430	4·9152	4·9894	5·0658	5·1446	11°	73	146	220	293	366	439	512	586	659
79°	5·1446	5·2257	5·3093	5·3955	5·4845	5·5764	5·6713	10°	88	175	263	350	438	526	613	701	788
80°	5·6713	5·7694	5·8708	5·9758	6·0814	6·1970	6·3138	9°									
81°	6·3138	6·4348	6·5606	6·6912	6·8269	6·9682	7·1154	8°									
82°	7·1154	7·2687	7·4287	7·5958	7·7704	7·9530	8·1443	7°									
83°	8·1443	8·3450	8·5555	8·7769	9·0098	9·2553	9·5144	6°									
84°	9·5144	9·7882	10·078	10·385	10·712	11·059	11·4301	5°									
85°	11·430	11·826	12·251	12·706	13·197	13·727	14·301	4°									
86°	14·301	14·924	15·605	16·350	17·169	18·075	19·081	3°									
87°	19·081	20·206	21·470	22·904	24·542	26·432	28·636	2°									
88°	28·636	31·242	34·368	38·188	42·964	49·104	57·290	1°									
89°	57·290	68·750	85·940	114·59	171·89	343·77	+∞	0°									
90°	+∞																
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

The differences change very rapidly here so that they cannot be tabulated.

The cotangent of a small angle of x′ or the tangent of 90°—x′ is very nearly equal to 3437·7 divided by x.

NATURAL COTANGENTS

TABLE IV
LOGARITHMIC SINES

	0′	10′	20′	30′	40′	50′	60′		Mean Differences 1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	—∞	7·46373	7·76475	7·94084	8·06578	8·16268	8·24186	89°	Differences vary so rapidly here that tabulation is impossible. For small angles of x minutes log sin x′ or log cos (90°—x′)=log x+4·46373.								
1°	8·24186	8·30879	8·36678	8·41792	8·46366	8·50504	8·54282	88°									
2°	8·54282	8·57757	8·60973	8 63968	8·66769	8·69400	8·71880	87°									
3°	8·71880	8·74226	8·76451	8·78568	8·80585	8·82513	8·84358	86°									
4°	8·84358	8 86128	8·87829	8·89464	8·91040	8 92561	8·94030	85°									
5°	8·94030	8·95450	8·96825	8·98157	8·99450	9·00704	9·01923	84°									
6°	9·01923	9·03109	9 04262	9·05386	9·06481	9·07548	9·08589	83°									
7°	9·08589	9·09606	9·10599	9·11570	9·12519	9·13447	9·14356	82°	96	192	288	384	480	576	672	768	864
8°	9·14356	9 15245	9·16116	9·16970	9·17807	9·18528	9·19433	81°	85	169	254	338	423	507	592	679	761
9°	9·19433	9·20223	9·20999	9·21761	9·22509	9·23244	9 23967	80°	76	151	227	302	378	453	529	604	680
10°	9·23967	9·24677	9·25376	9·26063	9·26739	9·27405	9·28060	79°	68	136	204	272	341	409	477	545	613
11°	·28060	·28705	·29340	·29966	·30582	·31189	·31788	78°	62	124	186	248	310	373	435	497	559
12°	·31788	·32378	·32960	·33534	·34100	·34658	·35209	77°	57	114	171	228	285	342	399	456	513
13°	·35209	·35752	·36289	·36819	·37341	·37858	·38368	76°	53	105	158	210	263	316	368	421	473
14°	·38368	·38871	·39369	·39860	·40346	·40825	·41300	75°	49	98	147	195	244	293	342	391	440
15°	9·41300	9·41768	9 42232	9·42690	9·43143	9·43591	9·44034	74°	46	91	137	182	228	273	319	364	410
16°	·44034	·44472	·44905	·45334	·45758	·46178	·46594	73°	43	85	128	171	213	256	299	341	384
17°	·46594	·47005	·47411	·47814	·48213	·48607	·48998	72°	40	80	120	160	201	241	281	321	361
18°	·48998	·49385	·49768	·50148	·50523	·50896	·51264	71°	38	76	113	151	189	227	264	302	340
19°	·51264	·51259	·51991	·52350	·52705	·53056	·53405	70°	36	71	107	143	179	214	250	285	321

20°	9·53405	9·53751	9·54093	9·54433	9·54769	9·55102	9·55433	69°	34	68	101	135	169	203	237	270	304
21°	·55433	·55761	·56085	·56408	·56727	·57044	·57358	68°	32	64	96	128	161	193	225	257	289
22°	·57358	·57669	·57978	·58284	·58588	·58889	·59188	67°	31	61	92	122	153	183	214	244	275
23°	·59188	·59484	·59778	·60070	·60359	·60646	·60931	66°	29	58	87	116	146	174	204	233	262
24°	·60931	·61214	·61494	·61773	·62049	·62323	·62595	65°	28	65	83	111	139	166	195	222	250
25°	9·62595	9·62865	9·63133	9·63398	9·63662	9·63924	9·64184	64°	27	53	80	106	133	159	186	212	239
26°	·64184	·64442	·64698	·64953	·65205	·65456	·65705	63°	25	51	76	102	127	152	178	203	229
27°	·65705	·65952	·66197	·66441	·66682	·66922	·67161	62°	24	49	73	97	122	146	170	194	219
28°	·67161	·67398	·67633	·67866	·68098	·68328	·68557	61°	23	47	70	93	117	140	163	186	210
29°	·68557	·68784	·69010	·69234	·69456	·69677	·69897	60°	22	45	67	89	112	134	156	179	201
30°	9·69897	9·70115	9·70332	9·70547	9·70761	9·70973	9·71184	59°	22	43	65	86	107	129	150	172	193
31°	·71184	·71393	·71602	·71809	·72014	·72218	·72421	58°	21	41	62	82	103	124	144	165	185
32°	·72421	·72622	·72823	·73022	·73219	·73416	·73611	57°	20	40	59	79	99	119	139	159	178
33°	·73511	·73805	·73997	·74189	·74379	·74568	·74756	56°	19	38	57	76	96	115	134	153	172
34°	·74756	·74943	·75128	·75313	·75496	·75678	·75859	55°	18	37	55	74	92	110	129	147	165
35°	9·75859	9·76039	9·76218	9·76395	9·76572	9·76747	9·76922	54°	18	35	53	71	89	106	124	142	159
36°	·76922	·77095	·77268	·77439	·77609	·77778	·77946	53°	17	34	51	68	86	103	120	137	154
37°	·77946	·78113	·78280	·78445	·78609	·78772	·78934	52°	17	33	50	66	83	99	116	132	149
38°	·78934	·79095	·79256	·79415	·79573	·79731	·79887	51°	16	32	48	64	80	95	112	127	143
39°	·79887	·80043	·80197	·80351	·80504	·80656	·80807	50°	15	31	46	62	77	92	108	123	138
40°	9·80807	9·80957	9·81106	9·81245	9·81402	9·81549	9·81694	49°	15	30	44	59	74	89	104	118	133
41°	·81694	·81839	·81983	·82126	·82269	·82410	·82551	48°	14	29	43	57	72	86	100	114	129
42°	·82551	·82691	·82830	·82968	·83106	·83242	·83378	47°	14	28	41	55	69	83	97	110	124
43°	·83378	·83513	·83648	·83781	·83914	·84046	·84177	46°	13	27	40	53	67	80	93	106	120
44°	·84177	·84308	·84437	·84566	·84694	·84822	·84949	45°	13	26	38	51	64	77	90	102	115
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COSINES

LOGARITHMIC SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	·984949	9·85074	9·85200	9·85324	9·85448	9·85571	9·85693	44°	12	25	37	50	62	74	87	99	112
46°	·85693	·85815	·85936	·86056	·86176	·86294	·86413	43°	12	24	36	48	60	72	84	96	108
47°	·86413	·86530	·86647	·86763	·86879	·86993	·87107	42°	12	23	35	46	58	70	81	93	104
48°	·87107	·87221	·87334	·87446	·87557	·87668	·87778	41°	11	22	34	45	56	67	78	89	100
49°	·87778	·87887	·87996	·88105	·88212	.88319	·88425	40°	11	22	32	43	54	65	76	86	97
50°	9·88425	9·88531	9·88636	9·88741	9·88844	9·88948	9·89050	39°	10	21	31	42	52	62	73	83	94
51°	·89050	·89152	·89254	·89354	·89455	·89554	·89653	38°	10	20	30	40	50	60	70	80	90
52°	·89653	·89752	·89849	·89947	·90043	·90139	·90235	37°	10	19	29	39	49	58	68	78	87
53°	·90235	·90330	·90424	·90518	·90611	·90704	·90796	36°	9	19	28	37	47	56	65	74	84
54°	·90796	·90887	·90978	·91069	·91158	·91241	·91336	35°	9	18	27	36	45	54	63	72	81
55°	9·91336	9·91425	9·91512	9·91599	9·91686	9·91772	9·91857	34°	9	17	26	35	44	52	61	70	78
56°	·91857	·91942	·92027	·92111	·92194	·92277	·92359	33°	8	17	25	34	42	50	59	67	76
57°	·92359	·92441	·92522	·92603	·92683	·92763	·92842	32°	8	16	24	32	41	49	57	65	73
58°	·92842	·92921	·92999	·93077	·93154	·93230	·93307	31°	8	16	23	31	39	47	55	62	70
59°	·93307	·93382	·93457	·93532	·93606	·93680	·93753	30°	8	15	23	30	37	45	32	60	67
60°	9·93753	9·93826	9·93898	9·93970	9·94041	9·94112	9·94182	29°	7	14	22	29	36	43	50	57	64
61°	·94182	·94252	·94321	·94390	·94458	·94526	·94593	28°	7	14	21	27	34	41	48	55	62
62°	·94593	·94660	·94727	·94793	·94858	·94923	·94988	27°	7	13	20	26	33	40	46	53	59
63°	·94988	·95052	·95116	·95179	·95242	·95304	·95366	26°	6	13	19	25	32	38	44	50	57
64°	·95366	·95427	·95488	·95549	·95609	·95668	·95728	25°	6	12	18	24	30	36	42	48	54

65°	9·95728	9·95786	9·95844	9·95902	9·95960	9.96017	9·96073	24°	6	12	17	23	29	35	40	46	52
66°	·96073	·96129	·96185	·96240	·96294	·96349	·96403	23°	6	11	17	22	28	33	38	44	50
67°	·96403	·96456	·96509	·96562	·96614	·96665	·96717	22°	5	10	16	21	26	31	36	42	47
68°	·96717	·96767	·96818	·96868	·96917	·96966	·97015	21°	5	10	15	20	25	29	34	40	44
69°	·97015	·97063	·97111	·97159	·97206	·97252	·97299	20°	5	9	14	19	24	28	33	38	42
70°	9·97299	9·97344	9·97390	9·97435	9·97479	9·97523	9·97567	19°	4	9	13	18	22	27	31	36	40
71°	·97567	·97610	·97653	·97696	·97738	·97779	·97821	18°	4	9	13	17	21	26	30	34	38
72°	·97821	·97861	·97902	·97942	·97982	·98021	·98060	17°	4	8	12	16	20	24	28	32	36
73°	·98060	·98060	·98136	·98174	·98211	·98248	·98284	16°	4	8	11	15	19	22	26	30	34
74°	.98284	·98320	·98356	.98391	·98426	·98460	.98494	15°	4	7	11	14	18	21	25	28	32
75°	9·98494	9·98528	9·98561	9·98594	9·98627	9·98659	9·98690	14°	3	7	10	13	17	20	23	26	30
76°	·98690	·98722	·98753	·98783	·98813	·98843	·98872	13°	3	6	9	12	15	18	21	24	27
77°	·98872	·98901	·98930	·98958	·98986	·99013	·99040	12°	3	6	8	11	14	17	20	22	25
78°	·99040	·99067	·99093	·99119	·99145	·99170	·99195	11°	3	5	8	10	13	16	18	21	23
79°	·99195	·99219	·99243	·99267	·99290	·99313	·99335	10°	2	5	7	9	12	14	16	19	21
80°	9·99335	9·99357	9·99379	9·99400	9·99421	9·99442	9.99462	9°	2	4	6	8	11	13	15	17	19
81°	·99462	·99482	·99501	·99520	·99539	·99557	·99575	8°	2	4	5	8	10	11	13	15	17
82°	·99575	·99593	·99610	·99627	·99643	·99659	·99675	7°	2	3	4	7	8	10	12	13	15
83°	·99675	·99690	·99705	·99720	·99734	·99748	·99761	6°	1	3	4	6	7	9	10	12	13
84°	·99761	·99775	·99787	·99800	·99815	·99823	·99834	5°	1	3	4	5	6	8	9	10	11
85°	9·99834	9·99845	9·99856	9·99866	9·99876	9·99885	9·99894	4°	1	2	3	4	5	6	7	8	9
86°	·99894	·99903	·99911	·99919	·99926	·99934	·99940	3°	1	2	2	3	4	5	5	6	7
87°	·99940	·99947	99953	·99959	·99964	·99969	·99974	2°	1	1	2	2	3	3	4	4	5
88°	·99974	·99978	·99982	·99985	·99988	·99991	·99993	1°	0	1	1	1	2	2	2	2	3
89°	·99993	·99995	·99997	·99998	·99999	10·00000	10·00000	0°									
90°	10·00000																
.	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8	9′

LOGARITHMIC COSINES

Table V
LOGARITHMIC TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	$-\infty$	7·46373	7·76476	7·94086	8·06581	8·16273	8·24192	89°	Differences vary so rapidly here that tabulation is impossible. For small angles of x minutes log tan x′ or log cot (90°−x′) = log x + 4·46373.								
1°	8·24192	8·30888	8·36689	8·41807	8·46385	8·50527	8·54303	88°									
2°	8·54308	8·57788	8·61009	8·64009	8·66816	8·69453	8·71940	87°									
3°	8·71940	8·74292	8·76525	8·78649	8·80674	8·82610	8·84464	86°									
4°	8·84464	8·86243	8·87953	8·89598	8·91185	8·92716	8·94195	85°									
5°	8·94195	8·95627	8·97013	8·98358	8·99662	9·00930	9·02162	84°									
6°	9·02162	9·03361	9·04528	9·05666	9·06775	9·07858	9·08914	83°									
7°	9·08914	9·09947	9·10956	9·11943	9·12909	9·13854	9·14780	82°	98	195	293	391	488	586	684	782	879
8°	9·14780	9·15688	9·16577	9·17550	9·18306	9·19146	9·19971	81°	87	173	260	346	433	519	606	692	779
9°	9·19971	9·20782	9·21578	9·22361	9·23130	9·23887	9·24632	80°	78	155	233	310	388	466	543	621	698
10°	9·24632	9·25365	9·26086	9·26797	9·27496	9·28186	9·28856	79°	71	141	212	282	354	420	494	564	635
11°	·28865	29535	·30195	·30846	·31489	·32122	·32747	78°	65	129	194	259	323	388	453	518	582
12°	·32747	·33365	·33974	·34576	·35170	·35757	·36336	77°	60	120	179	239	299	359	419	478	538
13°	·36336	·36909	·37476	·38035	·38589	·39136	·39677	76°	56	111	167	222	278	334	389	445	500
14°	·39677	·40212	·40742	·41266	·41784	·42297	·42805	75°	52	104	156	208	261	313	365	417	469
15°	9·42805	9·43308	9·43806	9·44299	9·44787	9·45271	9·45750	74°	49	98	147	196	245	294	243	392	442
16°	·45750	·46224	·46694	·47160	·47622	·48080	·48534	73°	46	93	139	186	232	278	325	371	418
17°	·48534	·48984	·49430	·49872	·50311	·50746	·51178	72°	44	88	132	176	220	264	308	352	396
18°	·51178	·51606	·52031	·52452	·52870	·53285	·53697	71°	42	84	126	168	210	252	294	336	378
19°	·53697	·54106	·54512	·54915	·55315	·55712	·56107	70°	40	80	121	160	201	241	281	321	362

20°	9·56107	9·56498	9·56887	9·57274	9·57658	9·58039	9·58418	69°	39	77	116	154	193	231	270	308	347
21°	·58418	·58794	·59168	·59540	·59909	·60276	·60641	68°	37	74	111	148	185	222	259	296	333
22°	·60641	·61004	·61364	·61722	·62079	·62433	·62785	67°	36	72	107	143	179	214	250	286	322
23°	·62785	·63135	·63484	·63830	·64175	·64517	·64858	66°	35	69	104	138	173	208	242	277	311
24°	·64858	·65197	·65535	·65870	·66204	·66537	·66867	65°	34	67	101	134	168	201	235	268	302
25°	9·66867	9·67196	9·67524	9·67850	9·68174	9·68497	9·68818	64°	33	65	98	130	163	195	228	260	293
26°	·68818	·69138	·69457	·69774	·70089	·70404	·70717	63°	32	63	95	126	158	190	221	253	284
27°	·70717	·71028	·71339	·71648	·71955	·72262	·72567	62°	31	62	92	123	154	185	216	246	277
28°	·72567	·72872	·73175	·73476	·73777	·74077	·74375	61°	30	60	90	130	151	181	211	241	271
29°	·74375	·74673	·74969	·75264	·75558	·75852	·76144	60°	29	59	88	118	147	177	206	236	265
30°	9·76144	9·76435	9·76725	9·77015	9·77303	9·77591	9·77877	59°	29	58	87	116	144	173	202	231	260
31°	·77877	·78163	·78448	·78732	·79015	·79297	·79579	58°	28	57	85	113	142	170	198	227	255
32°	·79579	·79860	·80140	·80419	·80697	·80975	·81252	57°	28	56	84	112	139	167	195	223	251
33°	·81252	·81528	·81803	·82078	·82852	·82626	·82899	56°	28	55	83	110	137	165	192	220	247
34°	·82899	·83171	·83442	·83713	·83984	·84254	·84523	55°	27	54	81	108	136	162	190	217	244
35°	9·84523	9·84791	9·85059	9·85327	9·85594	9·85860	9·86126	54°	27	54	80	107	134	160	188	214	241
36°	·86126	·86392	·86656	·86921	·87185	·87448	·87711	53°	26	53	79	106	132	158	285	212	238
37°	·87711	·87974	·88236	·88498	·88759	·89020	·89281	52°	26	52	78	105	131	157	183	209	236
38°	·89281	·89541	·89801	·90061	·90320	·90578	·90837	51°	26	52	78	104	130	156	182	208	234
39°	·90837	·91095	·91353	·91610	·91868	·92125	·92381	50°	26	52	77	103	129	155	180	205	232
40°	9·92381	9·92638	9·92894	9·93150	9·93405	9·93661	9·93916	49°	26	51	77	102	128	154	179	205	230
41°	·93916	·94171	·94426	·94681	·94935	·95190	·95444	48°	25	51	76	102	127	153	178	204	229
42°	·95444	·95698	·95952	·96205	·96459	·96712	·96966	47°	25	51	76	101	127	152	177	203	228
43°	·96966	·97219	·97472	·97725	·97978	·98231	·98484	46°	25	51	76	101	127	152	177	202	228
44°	·98484	·98737	·98989	·99242	·99495	·99747	10·00000	45°	25	51	76	101	127	152	177	202	228
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COTANGENTS

LOGARITHMIC TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	10·00000	10·00253	10·00505	10·00758	10·01011	10·01263	10·01516	44°	25	51	76	101	127	152	177	202	228
46°	·01516	·01769	·02022	·02275	·02528	·02781	.03034	43°	25	51	76	101	127	152	177	202	228
47°	·03034	·03288	·03541	·03795	·04048	·04302	·04556	42°	25	51	76	101	127	152	177	203	228
48°	·04556	·04810	·05065	·05319	·05574	·05829	·06084	41°	25	51	76	102	127	153	178	204	229
49°	·06084	·06339	·06594	·06850	·97106	·07362	·07619	40°	26	51	77	102	128	154	179	205	230
50°	10·07619	10·07875	10·08132	10·08390	10·08647	10·08905	10·09163	39°	26	52	77	103	129	155	180	206	232
51°	·09163	·09422	·09680	·09939	·10199	·10459	·10719	38°	26	52	78	104	130	156	182	208	234
52°	·10719	·10980	·11241	·11502	·11764	·12026	·12289	37°	26	52	78	105	131	157	183	209	236
53°	·12289	·12552	·12815	·13079	·13344	·13608	·13874	36°	26	53	79	106	132	158	185	212	238
54°	·13874	·14140	·14406	·14673	·14941	·15209	·15477	35°	27	54	80	107	134	160	188	214	241
55°	10·15477	10·15746	10·16016	10·16287	10 16558	10·16829	10·17101	34°	27	54	81	108	136	162	190	217	244
56°	·17101	·17374	·17648	·17922	·18197	·18472	·18748	33°	28	55	83	110	137	165	192	220	247
57°	·18748	·19025	·19303	·19581	·19860	·20140	·20421	32°	28	56	84	112	139	167	195	223	251
58°	·20421	·20703	·20985	·21268	·21552	·21837	·22123	31°	28	57	85	113	142	170	198	227	255
59°	·22123	·22409	·22[illegible]97	·22985	·23275	·23565	·23856	30°	29	58	87	116	144	173	202	231	260
60°	10·23856	10·24148	10·24442	10·24736	10·25031	10·25327	10·25625	29°	29	59	88	118	147	177	206	236	265
61°	·25625	·25923	·26223	·26524	·26825	·27128	·27433	28°	30	60	90	120	151	181	211	241	271
62°	·27433	·27738	·28045	·28352	·28661	·28972	·29283	27°	31	62	92	123	154	185	216	246	277
63°	·29283	·29596	·29911	·30226	·30543	·30862	·31182	26°	32	63	95	126	158	190	221	253	284
64°	·31182	·31503	·31826	·32150	·32476	·32804	·33133	25°	33	65	98	130	163	195	228	260	293

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
65°	10·33133	10·33463	10·33793	10·34130	10·34465	10·34803	10·35142	24°	34	67	101	134	168	201	135	268	302
6[illegible]	·35142	·35483	·35823	·36170	·36516	·36865	·37215	23°	35	69	104	138	173	208	242	277	311
6[illegible]	·37215	·37567	·37921	·38278	·38636	·38996	·39359	22°	36	72	107	143	179	214	251	286	322
68°	·39359	·39724	·40091	·40460	·40832	·41206	·41582	21°	37	74	111	141	185	222	259	296	333
69°	·41582	·41961	·42342	·42726	·43113	·43502	·43893	20°	3	77	116	154	993	231	270	309	347
70°	10·43893	·1044288	10·44685	10·45085	10·45488	10·45894	10·46303	19°	40	80	121	160	201	241	281	321	362
71°	·46303	·46715	·47130	·47548	·47969	·48394	·48822	18°	42	84	126	168	210	252	294	336	378
72°	·48822	·49254	·49689	·50128	·50570	·51016	·51466	17°	44	88	132	176	220	264	308	352	396
73°	·51466	·51920	·52378	·52840	·53306	·53776	·54250	16°	46	93	139	186	232	278	325	371	418
74°	·54250	·54729	55213	·55701	·56194	·56692	·57195	15°	49	98	147	196	245	294	343	392	442
75°	10·57195	10·57703	10·58216	10·58734	10·59258	10·59788	10·60323	14°	52	104	156	208	261	313	365	417	469
76°	·60323	·60864	·61411	·61965	·92524	·63091	·63664	13°	56	111	167	222	278	334	389	445	500
77°	·63664	·64243	·64830	·65424	·66026	·66635	·67253	12°	60	120	179	239	299	359	419	478	538
78°	·67253	·67878	·68511	·69154	·69805	·70465	·71135	11°	65	129	194	259	323	388	453	518	582
79°	·71135	·71814	·72504	73203	·73914	·74635	·75368	10°	71	141	212	282	354	420	494	564	635
80°	10·75368	10·76113	10·76870	10·77639	10·78422	10·79218	10·80029	9°	78	155	233	310	388	466	543	621	698
81°	·80029	·80854	·81694	·82550	·83423	·84312	·85220	8°	87	73	260	346	433	519	606	692	779
82°	·85220	·86146	·87091	·88057	·89044	·90053	·91086	7°	98	95	293	391	488	586	684	782	879
83°	·91086	·92142	·93225	·94334	·95472	·96639	·97830	6°									
84°	·97838	.99070	11·00338	11·01642	11·02987	11·04373	11·05805	5°									
85°	11·05805	11·07284	11·08815	11 10402	11·12047	11·13757	11·15536	4°									
86°	11·15536	11·17390	11·19326	11·21351	11·23475	11·25708	11·28060	3°									
87°	11·28060	11·30547	11·33184	11·35991	11·38991	11·42212	11·45602	2°	Differences vary so rapidly here that tabulation is impossible.								
88°	11·45692	11·49473	11·53615	11·58193	11·63311	11·69112	·1175808	1°									
89°	11·75808	11·83727	11·93419	12·05914	12·23524	12·53627	+ ∞	0°									
90°	+ ∞																

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SUBJECT INDEX